NOTES
ON
RANKINE'S
APPLIED MECHANICS.

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ON

RANKINE'S APPLIED MECHANICS.

BY

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INTRODUCTION.

The following pages are the result of putting in permanent form some of the matter which it has been found expedient or necessary to give by dictation to students in the Worcester Free Institute, who pursue for the first time, the study of Rankine's Applied Mechanics. The object in their publication is not to furnish a key, or provide a substitute for diligent study and careful thought on the part of the student, but rather to encourage him by giving such suggestions, solutions, and references as experience has shown that the average student requires; thus economizing time in the preparation of the lesson, and also giving the instructor opportunity to devote the time spent in the class room to recitations, and to the application of the principles and formulæ of the lesson, to practical problems.

To what may be strictly called notes on the "Applied Mechanics," I have added a brief explanation and illustration of the method of producing the reciprocal diagram of stresses, substantially taken from "Economics of Construction," by R. E. Bow, C. E. Also a separate treatise on strength of beams, and an investigation of a particular problem relating to seven bar parallel motions, known as "Peaucillier's Parallel Motion."

This work has been prepared from materials drawn from various sources, especially from notes given by Prof. Eustis, of The Lawrence Scientific School, to the class of '68.

I have also received assistance from George H. White, B. S., a graduate of the Free Institute, and have inserted on several articles of the Applied Mechanics, notes which are entirely his own work. I have endeavored to make proper reference to works from which quotations or extracts have been taken.

The blank pages at the end are intended to receive such supplementary notes as the instructor may find adapted to the needs and capacity of his class.

GEORGE I. ALDEN.

WORCESTER FREE INSTITUTE, Feb. 1st, 1877.

NOTE.—Some errors found in the earlier editions of "Rankine's Applied Mechanics" have been corrected in later editions, issued since these notes were prepared. References in the notes, to such errors, are allowed to remain for the benefit of those who may have copies of the older editions of the "Applied Mechanics."

INTEGRATION.

The following integrals are of frequent occurrence, and are here given for future reference :

(A) . $$\int d\,x\,\sqrt{a^2 - x^2}$$

In the general formula for integrating by parts, $\int u\,d\,v = u\,v - \int v\,.\,d\,u$, let $u = \sqrt{a^2 - x^2}$, and $d\,x = d\,v$; then $d\,u = -\frac{x\,d\,x}{\sqrt{a^2 - x^2}}$ and $x = v$

$$\therefore \int d\,x\,\sqrt{a^2 - x^2} = x\,\sqrt{a^2 - x^2} + \int \frac{x^2\,d\,x}{\sqrt{a^2 - x^2}}$$

Again,

$$\therefore \int d\,x\,\sqrt{a^2 - x^2} = \int \frac{d\,x\,(a^2 - x^2)}{\sqrt{a^2 - x^2}} = -\int \frac{x^2\,d\,x}{\sqrt{a^2 - x^2}} + \int \frac{a^2\,d\,x}{\sqrt{a^2 - x^2}}.$$

$$\therefore 2\int d\,x\,\sqrt{a^2 - x^2} = x\,\sqrt{a^2 - x^2} + \int \frac{a^2\,d\,x}{\sqrt{a^2 - x^2}}$$

$$\int \frac{a^2\,d\,x}{\sqrt{a^2 - x^2}} = \int a^2 \frac{d\left(\frac{x}{a}\right)}{\sqrt{1 - \frac{x^2}{a^2}}} = a^2 \sin^{-1}\frac{x}{a} + C.$$

$$\therefore 2\int d\,x\,\sqrt{a^2 - x^2} = x\,\sqrt{a^2 - x^2} + a^2\,\sin^{-1}\frac{x}{a} + C$$

$$\therefore \int d\,x\,\sqrt{a^2 - x^2} = \frac{x}{2}\,\sqrt{a^2 - x^2} + \frac{a^2}{2}\,\sin^{-1}\frac{x}{a} + C.$$

(B) $$\int \frac{d\,x}{\sqrt{a^2 + x^2}}$$

Let $\sqrt{a^2 + x^2} = z - x$: Then $a^2 + x^2 = z^2 - 2z,x + x^2$ or

$a^2 = z^2 - 2\,z\,x$. The differential of this equation is $0 = 2\,z\,d\,z - 2\,z\,d\,x - 2\,x\,d\,z \therefore d\,x = \dfrac{(z-x)\,d\,z}{z}$

$$\therefore \int \frac{d\,x}{\sqrt{a^2+x^2}} = \frac{(z-x)\,d\,z}{(z-x)\,z} = \int \frac{d\,z}{z} = \log_e z + C = \log_e (x + \sqrt{a^2+x^2}) + C$$

(C) $$\int d\,x\,\sqrt{a^2+x^2}$$

In the formula for integration by parts, $\int u\,d\,v = u\,v - \int v\,d\,u$, let $\sqrt{a^2+x^2} = u$ and $d\,x = d\,v$. Then $x = v$ and $\dfrac{x\,d\,x}{\sqrt{a^2+x^2}} = d\,u$

$$\therefore \int d\,x\,\sqrt{a^2+x^2} = x\,\sqrt{a^2+x^2} - \int \frac{x^2\,d\,x}{\sqrt{a^2+x^2}}.$$ Again,

$$\int d\,x\,\sqrt{a^2+x^2} = \int \frac{d\,x\,(a^2+x^2)}{\sqrt{a^2+x^2}} = \int \frac{a^2\,d\,x}{\sqrt{a^2+x^2}} + \int \frac{x^2\,d\,x}{\sqrt{a^2+x^2}}$$

$$\therefore 2 \int d\,x\,\sqrt{a^2+x^2} = x\,\sqrt{a^2+x^2} + a^2 \int \frac{d\,x}{\sqrt{a^2+x^2}}$$

By integral B, we have $a^2 \int \dfrac{d\,x}{\sqrt{a^2+x^2}} = a^2 \log_e (x + \sqrt{a^2+x^2}) + C$.

$$\therefore \int d\,x\,\sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log_e (x + \sqrt{a^2+x^2}) + C.$$

(D) $$\int x^2\,d\,x\,\sqrt{a^2-x^2}$$

In the formula $\int u\,d\,v = u\,v - \int v\,d\,u$, let $x\,d\,x\,\sqrt{a^2-x^2} = d\,v$, and $x = u$: then $v = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$ and $d\,u = d\,x$

$$\therefore \int x^2\,d\,x\,\sqrt{a^2-x^2} = -\frac{x}{3}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{3}\int d\,x\,(a^2-x^2)^{\frac{3}{2}}$$

$$= -\frac{x}{3}(a^2-x^2)^{\frac{3}{2}} + \frac{1}{3}\int d\,x\,(a^2-x^2)\,\sqrt{a^2-x^2}$$

$$= -\frac{x}{3}(a^2 - x^2)^{\frac{3}{2}} + \frac{1}{3}\int d\,x\,a^2\sqrt{a^2 - x^2}$$

$$-\frac{1}{3}\int x^2\,d\,x\,\sqrt{a^2 - x^2}$$

Transposing the last term of the last member, we have

$$\frac{4}{3}\int x^2\,d\,x\,\sqrt{a^2 - x^2} = -\frac{x}{3}(a^2 - x^2)^{\frac{3}{2}} + \frac{1}{3}\int a^2\,d\,x\sqrt{a^2 - x^2}$$

From integral A we have

$$\frac{1}{3}a^2\int d\,x\,\sqrt{a^2 - x^2} = \frac{a^2}{3}\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\} + \mathrm{C}.$$

$$\therefore \frac{4}{3}\int x^2\,d\,x\,\sqrt{a^2 - x^2} = -\frac{x}{3}(a^2 - x^2)^{\frac{3}{2}} + \frac{a^2\,x}{6}\sqrt{a^2 - x^2}$$

$$+\frac{a^4}{6}\sin^{-1}\frac{x}{a} + \mathrm{C}'.$$

$$\therefore \int x^2\,d\,x\,\sqrt{a^2 - x^2} = -\frac{x}{4}(a^2 - x^2)^{\frac{3}{2}} + \frac{a^2\,x}{8}\sqrt{a^2 - x^2}$$

$$+\frac{a^4}{8}\sin^{-1}\frac{x}{a} + \mathrm{C}.$$

This integral, taken between the limits a and 0, reduces to $\frac{a^4}{8}\sin^{-1}\frac{x}{a}$, which can easily be memorized.

(E) $$\int \cos^2\theta\,d\,\theta.$$

Integrals of the powers of $\sin\theta$ and $\cos\theta$ are found by substituting for these functions of θ, their values in terms of the multiple angles, as in the following case:

$$\int \cos^2\theta\,d\,\theta = \int \frac{1 + \cos 2\,\theta}{2}\,d\,\theta = \frac{\theta}{2} + \frac{\sin\,2\,\theta}{4} + \mathrm{C}.$$

Article 83.*

In solving the following problems the student should always sketch a figure representing the surface or solid under consideration, and one of the elementary parts into which it is conceived to be divided, and determine the limits for the integral by an inspection of this figure. In double or triple integrations, we

Note.—The first three integrals are taken from Todhunter's Integral Calculus.
Note.—The abbreviation A. M. will be used for Rankine's Applied Mechanics.
* The reference is to Article 83 *of the Applied Mechanics.*

must, in general, first integrate with respect to one of the variables, *between the proper limits*, and express the result in terms of the other variables. This may be continued until the complete integral is obtained.

To illustrate this process take the general formulæ for center of gravity of a solid, viz:

$$x_0 = \frac{\int\int\int x\, dx\, dy\, dz}{\int\int\int dx\, dy\, dz}$$ and similar values for y_0 and z_0.

The following application of these formulæ is found in Todhunter's Analytical Statics.

Problem. Find the center of gravity of the eighth part of an ellipsoid cut off by the three principal planes.

Let Fig. (1) represent the solid in question, the equation of the surface being $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ (1)

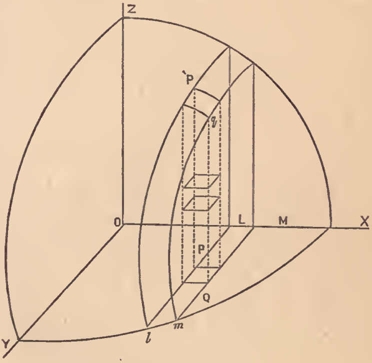


Fig. 1.

If we integrate first with respect to z, between the limits z_1 and zero, we include all the elements $(dx\, dy\, dz)$ in the prism P Q. Next integrate with respect to y between the limits y_1 and zero. We thus include all the prisms in the slice between the planes l L and m M.

From Equation (1), $z_1 = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$; from Equation of ellipse in the plane X Y,

$$y_1 = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$\text{Thus } x_0 = \frac{\int_o^a \int_o^{y_1} \int_o^{z_1} x \, dx \, dy \, dz}{\int_o^a \int_o^{y_1} \int_o^{z_1} dx \, dy \, dz} = \frac{\int_o^a \int_o^{y_1} x \, dx \, dy \, z_,}{\int_o^a \int_o^{y_1} dx \, dy \, z_,}$$

$$= \frac{\int_o^a \int_o^{y_1} c \, x \, dx \, dy \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}{\int_o^a \int_o^{y_1} c \, dx \, dy \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} = \frac{\int_o^a \int_o^{y_1} \frac{1}{b} \, x \, dx \, dy}{\int_o^a \int_o^{y_1} \frac{1}{b} \, dx \, dy}$$

$$\frac{\sqrt{y_,^2 - y^2}}{\sqrt{y_,^2 - y^2}} = \frac{\int_o^a x \, dx \frac{\pi \, y_,^2}{4}}{\int_o^a dx \, \frac{\pi \, y_,^2}{4}} = \frac{\int_o^a x \, dx \frac{b^2}{a^2} (a^2 - x^2)}{\int_o^a dx \frac{b^2}{a^2} (a^2 - x^2)} = \frac{3}{8} \, a.$$

The other coördinates of the center of gravity will easily be found by taking moments with reference to the axes OX and OZ, and following the integration as above indicated.

Problem III. Draw a line through C parallel to E B, forming a parallelogram and a triangle. A line joining C and the center of gravity of the triangle will be parallel to O D.

Let $\angle$ BOD $= \theta$. Then equating the moment of the trapezoid, about A B, to the moment of this parallelogram and triangle, about the same line, we have

$$\frac{B + b}{2}. \ \overline{O\,D}. \sin\theta \, . \, x_0 \sin\theta \, . \, w = b \, \overline{OD} \sin\theta \, . \, \frac{\overline{O\,D}}{2} \sin\theta \, . \, w \, +$$

$$(B - b) \, \frac{\overline{O\,D}}{2} \sin\theta . \, \frac{\overline{O\,D}}{3} \sin\theta \, . \, w; \quad \frac{B + b}{2} \, . \, x_0 = \frac{\overline{O\,D}}{2} \left(\frac{B - b}{3} + b. \right)$$

$$\therefore \, x_0 = \frac{\overline{O\,D}}{2} \left(\frac{2\,(B - b) + 6\,b}{3\,(B + b)} \right) = \frac{\overline{O\,D}}{2} \left(\frac{2\,B + 4\,b}{3\,(B + b)} \right) = \frac{\overline{O\,D}}{2}$$

$$\left(\frac{3\,(B + b) - (B - b)}{3\,(B + b)} \right) = \frac{\overline{O\,D}}{2} \left(1 - \frac{B - b}{3\,(B + b)} \right).$$

Problem V. Taking moments about O Y we have for the arm of the element $y dx$, the distance x . sin Y O X. The distance of the center of gravity from O Y on a line at right angles to O Y is x_0 . sin Y O X

$$\therefore \, x_0 \sin \text{Y O X} = \sin \text{Y O X} \frac{\int_o^{x_1} x \, y \, dx}{\int_o^{x_1} y \, dx}; \quad x_0 = \frac{\int_o^{x_1} x \, y \, dx}{\int_o^{x_1} y \, dx}.$$

$$y_0 = \frac{\int_o^{x_1} \frac{y}{2} y \, dx}{\int_o^{x_1} y \, dx} = \frac{\int_o^{x_1} y^2 \, dx}{2 \int_o^{x_1} y \, dx}.$$

The equation of the curve is $y^2 = 2 p^1 x$.

PROBLEM VII. In this, as in many problems relating to the circle, it is convenient to use polar coördinates. Let O (Fig. 32 A. M.) be the pole, ρ the radius vector, and ϕ the variable angle measured from the axis O X. The circle is divided into differential areas, by concentric circles $d\rho$ apart, and by radii making an angle $d\phi$ with each other.

Then a differential area $= \rho \, d \, \phi \, . \, d \, \rho$.
Its distance from O Y $= \rho \cos \phi$.
Its moment about O Y $= w \, \rho^2 \, d \, \rho \, . \, \cos \phi \, d \, \phi$.

$\therefore$ in this case

$$x_0 = \frac{w \int_0^R \int_{-\phi}^{+\phi} \rho^2 \, d \, \rho \, . \cos \phi \, . \, d \, \phi}{w \int_0^R \int_{-\phi}^{+\phi} \rho \, d \, \rho \, . \, d \, \phi} = \frac{\frac{R^3}{3} \int_{-\phi}^{+\phi} \cos \phi \, d \, \phi}{\frac{R^2}{2} \int_{-\phi}^{+\phi} d \, \phi}$$

$$= \frac{2}{3} R \frac{(\sin \phi)_{-\phi}^{+\phi}}{(\phi)_{-\phi}^{+\phi}} = \frac{2}{3} R \left(\frac{\sin \phi - \sin (-\phi)}{2 \, \phi} \right) = \frac{2}{3} R \frac{\sin \phi}{\phi}.$$

PROBLEM VIII. $$x_0 = \frac{\int_{r \cos \theta}^{r} xy \, dx}{\int_{r \cos \theta}^{r} y \, d \, x} = \frac{\int_{r \cos \theta}^{r} x (r^2 - x^2)^{\frac{1}{2}} \, dx}{\int_{r \cos \theta}^{r} (r^2 - x^2)^{\frac{1}{2}} \, d \, x}$$

$$y_0 = \frac{\int_0^y \int_{r \cos \theta}^{r} y \, d \, y \, d \, x}{\int_0^y \int_{r \cos \theta}^{r} d \, y \, d \, x} = \frac{\int_{r \cos \theta}^{r} \frac{1}{2} (r^2 - x^2) \, d \, x}{\int_{r \cos \theta}^{r} (r^2 - x^2)^{\frac{1}{2}} \, d \, x}.$$

PROBLEM XVI. Take moments about O Y. The moment of the annular wedge is equal to the moment of the whole wedge, minus the moment of the interior wedge. (See Art. 76 A. M.) Compare with Problem 15.

PROBLEMS XIX and XX. Conceive a plane passing through O X, (Fig. 39, A. M.) making a variable angle γ with the plane X Y. Let ρ be a radius vector in this plane, making a variable angle ϕ with O X.

Then $\rho \, d \, \phi \, d \, \rho$ is an elementary area in the above described plane. When that plane revolves about O X, through the angle

$d\gamma$, this elementary area describes a volume which is measured by the area times the distance through which it moves. This distance is the arc of a circle whose radius is $\rho \sin \phi$, and angle $d\gamma$, and $= \rho \sin \phi d\gamma$.

$\therefore$ Elementary volume $= \rho d\phi \,.\, d\rho \,.\, \rho \sin \phi d\gamma$.

Its distance from the plane Y Z $= \rho \cos \phi$; O Z being the third rectangular axis.

$$x_0 = \frac{\int\int\int \rho \, d\phi \,.\, d\rho \,.\, \rho \sin\phi \, d\gamma \,.\, \rho \cos\phi \,.\, w}{\int\int\int \rho \, d\phi \,.\, d\rho \,.\, \rho \sin\phi \, d\gamma \,.\, w}$$

$$= \frac{\int\int\int \rho^3 \, d\rho . \sin\phi \cos\phi \, d\phi . \, d\gamma}{\int\int\int \rho^2 \, d\rho . \sin\phi \, d\phi . \, d\gamma}.$$

Integrating between proper limits, this formula will give the value of x_0 for these, and similar problems.

In problem XIX the limits are $\rho \Big\}^{r}_{r'}$; $\phi \Big\}^{\beta}_{\alpha}$; $\gamma \Big\}^{2\pi}_{0}$

" " XX " " $\rho \Big\}^{r}_{r'}$ $\phi \Big\}^{\frac{\pi}{2}}_{0}$; $\gamma \Big\}^{+\theta}_{-\theta}$

In problem XX, and in similar problems,

$$y_0 = \frac{\int\int\int \rho \, d\phi \,.\, d\rho \,.\, \rho \sin\phi \, d\gamma \,.\, \rho \sin\phi \cos\gamma \,.\, w}{\int\int\int \rho \, d\phi \,.\, d\rho \,.\, \rho \sin\phi \, d\gamma \,.\, w}$$

The limits for this integral, in problem XX, are as stated above.

Article 91.

The expressions for moments around O X, O Y, and O Z, on page 74, are easily obtained by resolving the force P, having its point of application in the plane X Y, into components parallel to the three axes, and taking the moments of these components. Equation 4 is reduced by the relation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. See Church's Analytical Geometry, Art. 48, Equa. 4.

Equation 5 A has for its first member the sum of the rectangles of the cosines of the angles which two lines make with three rectangular axes, which is equal to the cosine of the angle between the two lines. See Church's Analytical Geometry, Art. 48, Equa. 5. If this cosine is 0, the angle must be 90°. The equation is verified by multiplying the first of equations 5

by cos α, the second by cos β, and the third by cos γ, and adding the products. The second member of this sum must then be reduced by introducing the values of M_1, M_2, M_3, from Equation 3.

ARTICLE 95.

Equations 3.—The point whose coördinates are expressed by this equation, is situated in the angle $Y^1 \, O \, X^1$.

PROBLEM.

Find the moment of inertia of a rectangle of length h and breadth b, about a neutral axis perpendicular to its plane. See Equation 5.

$$\text{Ans. } \frac{b\,h}{12}\left(b^2 + h^2\right).$$

Equations 13.—Complete figure 45, A. M., by drawing a line from C, parallel to O T, and call the point where this line intersects the line $O\,X_1$, N. From N, draw a line parallel to T C, and call its point of intersection with O T, H. The angle $T\,O\,X_1 = 90° - \angle X_1\,O\,Y^1 = \beta^1$.

Take the most general form of the equation of a straight line, $A\,x + B\,y + C = 0$, and write it in the form $\frac{A}{C}\,x + \frac{B}{C}\,y + 1 = 0$. Since this equation is true for all values of x and y, it will be true when x or y becomes 0.

Let the intercept on the axis $O\,X = c$, and the intercept on $O\,Y = d$, so that, when $y = 0$, $x = c$, and when $x = 0$, $y = d$.

Make $y = 0$. Then $\frac{A}{C}\,c + 1 = 0$, or $\frac{A}{C} = -\frac{1}{c}$

Make $x = 0$. Then $\frac{B}{C}\,d + 1 = 0$, or $\frac{B}{C} = -\frac{1}{d}$.

These values introduced into the general equation, reduce it to $-\frac{x}{c} - \frac{y}{d} + 1 = 0$.

$$\text{or } \frac{x}{c} + \frac{y}{d} = 1. \qquad \text{(A)}$$

Now multiply both members of this equation by n, drawn perpendicular to the straight line, and making the angle β^1 with the axis O X. See line O T, fig. 45, A. M.

Then $\frac{n}{c}x + \frac{n}{d}y = n$. But $\frac{n}{c} = \cos\beta^1$ and $\frac{n}{d} = \sin\beta^1$

$\therefore x\cos\beta^1 + y\sin\beta^1 = n.$ (B)

The equation of a tangent to an ellipse (T C) is $a^2 y y^1 + b^2 x x^1 = a^2 b^2$ (see Olney's General Geometry, Art. 136)

$\frac{x x^1}{a^2} + \frac{y y^1}{b^2} = 1$. Multiply this equation by n.

$$x\frac{x^1 n}{a^2} + y\frac{y^1 n}{b^2} = n. \quad \text{(C)}$$

By comparing equations C and B, it will be seen that $\cos\beta^1$, $= \frac{x^1 n}{a^2}$, and $\sin\beta^1 = \frac{y^1 n}{b^2}$

$$\therefore x^1 = \frac{a^2\cos\beta^1}{n}, \text{ and } y^1 = \frac{b^2 . \sin\beta^1}{n}. \quad \text{(D)}$$

Substitute these values in equation B, for x and y.

$\frac{a^2 . \cos^2\beta^1 + b^2 . \sin^2\beta^1}{n} = n$; $n^2 = a^2\cos^2\beta^1 + b^2 . \sin^2\beta^1$ (E)

To find the value of $n\,t$.

The general equation of a normal to an ellipse is $y - y^1 = \frac{a^2 y^1}{b^2 x^1}(x - x^1)$. (See Olney's General Geometry, Art. 156).

In this equation make $y = 0$, and divide both members by y^1.

$-1 = \frac{a^2}{b^2 x^1}(x - x^1)$; $\therefore -b^2 x^1 = a^2 x - a^2 x^1$; $x = \frac{(a^2 - b^2)x^1}{a^2} = \text{O N}.$

From Equation D, we have $x^1 = \frac{a^2 . \cos\beta^1}{n}$.

$\therefore \text{O N} = \frac{(a^2 - b^2)\cos\beta^1}{n}$

$\text{O N} . \sin\beta^1 = \text{N H} = \text{C T} = t = \frac{(a^2 - b^2)\cos\beta^1 . \sin\beta^1}{n}$

$$\therefore n\,t = (a^2 - b^2)\cos\beta^1 . \sin\beta^1.$$

Problem IV.

Use polar coördinates. An elementary area $= \rho\, d\phi . d\rho$. (See Note on Problem VII of Art. 83.) Its distance from

$\overline{OY} = \rho \cos \phi$. Its moment of inertia $= \rho\, d\phi\, d\rho\, (\rho \cos \phi)^2$

$\therefore I = \int_0^r \int_0^{2\pi} \rho^3\, d\rho .\ \cos^2 \phi\, d\phi = \frac{\pi r^4}{4} = \frac{\pi h^4}{64}$ (See Integral E).

Article 102.

Equations 1 and 2 are given in Chauvenet's Trigonometry, page 163, formula 53, and page 162, formula 48.

Article 105.

Problem II.—According to the preceding notation, the formulæ near the top of page 91 A. M. should be

$$O D = p_{xx} . \text{area } O B C + p_{yx} . \text{area } O C A + p_{zx} . \text{area } O A B.$$
$$O E = p_{xy} . \text{area } O B C + p_{yy} . \text{area } O C A + p_{zy} . \text{area } O A B.$$
$$O F = p_{xz} . \text{area } O B C + p_{yz} . \text{area } O C A + p_{zz} . \text{area } O A B.$$

Articles 110 and 111.

The student should carefully study these articles, and memorize the theorems, in order to understand Art. 112.

Article 112.

Equation 1.—From figure 54, A. M., we have

$$\overline{OR}^2 = \overline{OM}^2 + \overline{MR}^2 + 2\, \overline{OM} . \overline{MR} . \cos RMN$$

$$\text{or } p_r^2 = \left(\frac{p_x + p_y}{2}\right)^2 + \left(\frac{p_x - p_y}{2}\right)^2 + 2\left(\frac{p_x^2 - p_y^2}{4}\right) \cos 2\, \widehat{x\, n}$$

$$= \frac{p_x^2 + p_y^2}{2} + \frac{p_x^2 - p_y^2}{2} \cos 2\, \widehat{x\, n} = \frac{p_x^2}{2}\,(1 + \cos 2\, \widehat{x\, n})$$

$$+ \frac{p_y^2}{2}\,(1 - \cos 2\, \widehat{x\, n}).$$

But $\cos 2\, \widehat{x\, n} = 2 \cos^2 \widehat{x\, n} - 1 = 1 - 2 \sin^2 \widehat{x\, n}$.

$$\therefore p_r^2 = p_x^2 \cos^2 \widehat{x\, n} + p_y^2 \sin^2 \widehat{x\, n}$$

$$\therefore p_r = \sqrt{\left\{ p_x^2 . \cos^2 \widehat{x\, n} + p_y^2 \sin^2 \widehat{x\, n} \right\}} = O R. \qquad (1)$$

Equation 2.—From the triangle M O R, we have

$$\sin N O R : \sin R M O :: \overline{MR} : \overline{OR}$$

or $\sin \text{N O R} : \sin (180 - 2\,\widehat{x\,n}) :: \frac{p_x - p_y}{2} : p_r$

$$\therefore \sin \text{N O R} = \sin \widehat{n\,r} = \sin 2\,\widehat{x\,n}\,\frac{p_x - p_y}{2\,p_r}$$

$$\text{or N O R} = \widehat{n\,r} = \text{arc}\,.\,\sin\left(\sin 2\,\widehat{x\,n}\,\frac{p_x - p_y}{2\,p_r}\right). \qquad (2)$$

Proof that the locus of the point R is an ellipse whose semi-axes are p_x and p_y.

From the construction of figure 54, of the Applied Mechanics, it is readily seen that since OM = MQ = MP, the line

$$\text{PR} = \text{PM} - \text{MR} = \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2} = p_y.$$

$$\text{Also QR} = \text{QM} + \text{MR} = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} = p_x.$$

$$\text{The angle OQM} = 90^\circ - \widehat{xn}.$$

Therefore for the coördinates of the point R we have

$$x = \text{QR} \cos \widehat{xn} = p_x \cos \widehat{xn}.$$

$$y = \text{PR} \sin \widehat{xn} = p_y \sin \widehat{xn}.$$

Multiplying the first of these equations by p_y, the second by p_x; squaring and adding we have

$$p_y^2\,x^2 + p_x^2\,y^2 = p_x^2\,p_y^2$$

which is the equation of an ellipse with semiaxes p_x and p_y.

Equations 3 *and* 4.

$$p_n = \overline{\mathrm{OM}} - \overline{\mathrm{MR}} \cos (180^\circ - 2\, \widehat{xn}) = \frac{p_x + p_y}{2} + \frac{p_x - p_y}{2} \cos 2\, \widehat{xn}$$

$$= \frac{p_x}{2}(1 + \cos 2\, \widehat{xn}) + \frac{p_y}{2}(1 - \cos 2\, \widehat{xn}) = p_x \cos^2 \widehat{xn} + p_y \sin^2 \widehat{xn}.$$

$$p_t = \mathrm{MR} \sin 2\, \widehat{xn} = \frac{p_x - p_y}{2} 2 \sin \widehat{xn} \cos \widehat{xn} = (p_x - p_y) \sin \widehat{xn} \cos \widehat{xn}.$$

Comparison of figures 54 and 57, A. M.

In figure 54 suppose a second plane making an angle with the plane A B, to be drawn through O.

Let the same construction be made to represent the components of the principal stresses on this plane, as is made for AB.

The distance set off on the normal to this supposed plane, must be equal to OM, and the line corresponding to MR, must be equal to MR. There will also be a line corresponding to O R. Now suppose this second system of lines to be revolved about O, until the plane coincides with AB. The result will be ORMR′O of figure 57. By the construction of figure 57 it will be seen that MR = MR′ and OM is the same for both systems of lines, hence the figure satisfies the condition of the problem; viz : that $\mathrm{OM} = \frac{p_x + p_y}{2}$ and $\mathrm{MR} = \frac{p_x - p_y}{2}$.

Equation 15.—From the triangles OMR and OMR′ figure 57.

$$\overline{\mathrm{MR}}^2 = \overline{\mathrm{OM}}^2 + p^2 - 2\, p\overline{\mathrm{OM}} \cos \widehat{nr}.$$

$$\overline{\mathrm{MR}'}^2 = \overline{\mathrm{OM}}^2 + p'^2 - 2\, p'\, \mathrm{OM} \cos \widehat{n'r'}.$$

$$\therefore 0 = p^2 - p'^2 - 2.\, \mathrm{OM}\, (p. \cos \widehat{nr} - p' \cos \widehat{n'r'}).$$

$$\therefore \mathrm{OM} = \frac{p^2 - p'^2}{2\, (p. \cos \widehat{nr} - p' \cos \widehat{n'r'})}. \qquad (15)$$

Equations 16, are found by substituting $\frac{p_x + p_y}{2}$ for $\overline{OM}$ in the above values for MR, and MR′.

Equations 17.—For the first of these equations, draw a line from R perpendicular to ON (Fig. 57) and call its intersection with ON, A.

Then $\cos 2\,\widehat{xn} = \cos \mathrm{NMR} = \frac{\mathrm{MA}}{\mathrm{MR}}$. $\qquad \mathrm{MR} = \frac{p_x - p_y}{2}$.

$$\mathrm{MA} = \mathrm{OA} - \mathrm{OM} = \mathrm{OR} \cos \mathrm{NOR} - \mathrm{OM} = p \cos \widehat{nr} - \frac{p_x + p_y}{2}$$

$$= \frac{2\,p \cos \widehat{nr} - p_x - p_y}{2}.$$

$$\therefore \cos 2\,\widehat{xn} = \frac{2\,p\,.\cos \widehat{nr} - p_x - p_y}{p_x - p_y}.$$

The value of $\cos 2\,\widehat{xn'}$ is found by drawing a perpendicular to ON, from R′, and repeating the above process.

Equation 27.

From equation 25, $p + p' = \dfrac{2\sqrt{p\,p'}.\cos \widehat{nr}}{\cos \phi}$. (A)

Write equation 25 in the form

$\cos^2 \phi\,(p^2 + 2\,p\,p' + p'^2) = 4\,p\,p' \cos^2 \widehat{nr}$, suotract $4\,p\,p^1 \cos^2 \phi$ from both members of the equation, and extract the square root.

Then $p - p' = \dfrac{2\sqrt{p\,p'}.\sqrt{\cos^2 \widehat{nr} - \cos^2 \phi}}{\cos \phi}$. (B)

From equations A and B, we find

$$p = \frac{\sqrt{p\,p'}\,(\cos \widehat{nr} + \sqrt{(\cos^2 \widehat{nr} - \cos^2 \phi)})}{\cos \phi}.$$

$$p' = \frac{\sqrt{p\,p'}\,(\cos \widehat{nr} - \sqrt{(\cos^2 \widehat{nr} - \cos^2 \phi)})}{\cos \phi}.$$

$$\therefore \frac{p'}{p} = \frac{\cos \widehat{nr} - \sqrt{(\cos^2 \widehat{nr} - \cos^2 \phi)}}{\cos \widehat{nr} + \sqrt{(\cos^2 \widehat{nr} - \cos^2 \phi)}}. \qquad (27)$$

Article 124.

Formula for the center of pressure of a fluid, upon a plane surface.

In figure 62, let BF be the plane and suppose FO to be a line representing the continuation of this plane to O, the surface of the fluid. Let OB be the axis of x, and the axis of y at right angles to it.

Let the angle that OB makes with the surface of the fluid be represented by θ.

Let F be the origin of coördinates, and x be measured positively downward.

$dxdy =$ differential portion of this plane.

The pressure on a unit's surface of this plane is normal to the surface, and measured by $p = w\,(x + \text{OF}) \sin\theta$, in which w is the weight of a unit volume of the fluid. (See Art. 110, A. M.)

Let $\text{OF} = a$. Then $p = w\,(a + x)\sin\theta$.

By formula 2 of Art. 89, A. M.,

$$x_0 = \frac{\int_{-y}^{+y}\int_{o}^{\text{FB}} w\,x\,(a+x)\sin\theta\,.\,dxdy}{\int_{-y}^{+y}\int_{o}^{\text{FB}} w\,(a+x)\sin\theta\,dxdy}$$

$$= \frac{\int_{-y}^{+y}\int_{o}^{\text{FB}} (ax + x^2)\,dxdy}{\int_{-y}^{+y}\int_{o}^{\text{FB}} (a+x)\,dxdy}. \qquad \text{(A)}$$

$$y_0 = \frac{\int_{-y}^{y}\int_{o}^{\text{FB}} dxdy\;y\,(a+x)}{\int_{-y}^{y}\int_{o}^{\text{FB}} dxdy\,(a+x)}.$$

If the plane surface is symmetrical with respect to the axis x, the above equations become

$$x_0 = \frac{\int_{o}^{\text{FB}} y\,(ax + x^2)\,dx}{\int_{o}^{\text{FB}} y\,(a+x)\,dx}. \qquad y_0 = 0. \qquad \text{(B)}$$

If the plane extends to the surface of the fluid, a becomes zero, and the equations B reduce to $x_0 = \dfrac{\int_o^h x^2\,ydx}{\int_o^h xydx}$, h being the depth of the plane surface. (C)

PROBLEM 1.—Find the position of the center of pressure on a rectangular plane surface whose depth is h, and breadth b, when its upper edge is parallel to the surface of the fluid, and at a distance a below it.

From equation B we have *

$$x = \frac{\int_0^h y\,(a\,x + x^2)\,d\,x}{\int_0^h y\,(a + x)\,d\,x}. \quad \text{But } y = \frac{1}{2}\,b$$

$$\therefore x_0 = \frac{\int_0^h (a\,x + x^2)\,d\,x}{\int_0^h (a + x)\,d\,x} = \frac{\left(\frac{a\,x^2}{2} + \frac{x^3}{3}\right)_0^h}{\left(a\,x + \frac{x^2}{2}\right)_0^h} = \frac{3\,a\,h^2 + 2\,h^3}{6\,a\,h + 3\,h^2}.$$

PROBLEM 2.—Find the centers of pressure of the following surfaces, when their bases are in the surface of the fluid. (The depth of each surface $= h$.) Rectangle, triangle, and parabola.

Ans. $x_0 = \frac{2}{3}\,h, \frac{1}{2}\,h, \frac{4}{7}\,h$, respectively.

PROBLEM 3.—The parabola with its vertex in the surface, and base parallel to it.

Ans. $x_0 = \frac{5}{7}\,h.$

PROBLEM 4.—Find the center of pressure on a triangular surface whose depth is h, base parallel to the surface of the fluid, vertex upward and at a distance a below the surface of the fluid.

Ans. $x_0 = \frac{4\,a\,h^2 + 3\,h^3}{6\,a\,h + 4\,h^2}$ = distance from vertex.

ARTICLE 159.

Equations 4.—The load $\frac{w}{4}$ at 5, is resolved into two components, $\frac{w}{4}\cos i$, which is transmitted through 5 4 and produces a stress on all the pieces of the secondary truss 1 5 3 4, and $\frac{w}{4}\sin i$, one-half of which is supposed to cause a pull on the piece 1 5, and one-half a thrust on the piece 5 3.

Equations 4 are obtained by computing the stresses due to a

* NOTE.—It will be noticed that x_0 is measured downward from the top of the plane, *and not from the surface of the fluid.* Also that a is measured on the axis of x.

force $\frac{w}{4}$ cos i acting in the direction 54, and combining these stresses with the component $\frac{w}{4}$ sin i above mentioned.

Draw a line parallel to 54 and equal to $\frac{w}{4}$ cos i. From its extremities, draw lines parallel to 14 and 34, and from their intersection draw a line parallel to 13, to meet the first line, to which it is perpendicular.

Then from this diagram $R_{43} = R_{41} = \frac{w}{8}$ cos i, cosec $i = \frac{w}{8}$ cotan i.

$$R_{35} = \frac{w}{8} \cos i \operatorname{cotan} i \text{ (from diagram)} + \frac{1}{8} w \sin i \text{ (see above)}$$
$$= \frac{w}{8}\left(\frac{\cos^2 i + \sin^2 i}{\sin i}\right) = \frac{w}{8} \operatorname{cosec} i.$$

$$R_{51} = \frac{w}{8} \cos i \operatorname{cotan} i \text{ (from diagram)} - \frac{1}{8} w \sin i \text{ (see above)}$$
$$= \frac{w}{8}\left(\frac{1 - \sin^2 i - \sin^2 i}{\sin i}\right) = \frac{w}{8}\left(\frac{1 - 2\sin^2 i}{\sin i}\right) = \frac{w}{8} (\operatorname{cosec} i - 2 \sin i).$$

The Method of Drawing Reciprocal Diagrams.

Since the forces acting through any joint of a truss must be in equilibrium, these forces may be represented by a closed polygon whose sides are parallel to their directions.

A complete diagram for a truss under a given load ought, therefore, to contain as many such polygons as there are joints in the structure. In most cases a variety of diagrams can be drawn, any one of which will correctly represent the stresses in a given truss. The one that fulfills most nearly the condition, that for each joint in the truss, there shall be in the diagram a closed polygon whose sides are parallel to the forces acting through that joint, *and taken in the same order as those forces*, has been called the reciprocal diagram.

A method of notation given by R. H. Bow, C. E., renders the construction of reciprocal diagrams very easy in most cases. This method consists in placing upon the figure of the truss, a letter in each of the angular spaces formed by the intersection of the forces at each joint (a single letter in an enclosed area answering for all the internal angles formed by its sides), and

in the diagram the same letter at the junction of two lines representing concurring forces, that is found in the angle between those forces upon the figure of the truss.

To illustrate the application of this notation, let us construct the diagram for the frame represented by Fig. 78 A. M., on the supposition that it is loaded with W uniformly distributed over 12 and 13.

1st. Distribute the load upon the joints, as directed in Art. 147 A. M., or for a uniformly distributed load by placing upon each joint half of the load between the adjacent joints. This distribution gives $\frac{1}{6}$ W at the joints 4. 6. 1, 8 and 10, and $\frac{1}{12}$ W at the joints 2 and 3.

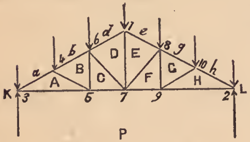


Fig. 1.

2d. Represent the directions of the external forces by lines, as in the accompanying Fig. 1. The forces are all given in direction, the internal forces or stresses being represented in direction by the pieces of the frame.

3d. Letter the angular spaces formed by concurring forces.

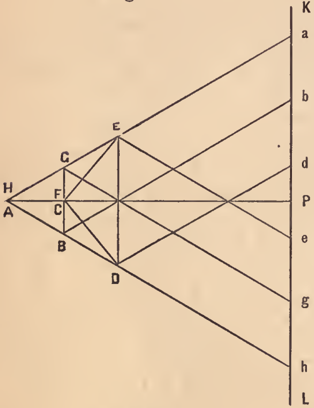


Fig. 2.

In Fig. 1, P occupies the space between 35 and the left hand support, 3 5 and 5 7, 5 7 and 7 9, 7 9 and 9 2, and 9 2 and the right hand support. K occupies the angular space at the left of the left hand support, between it and the load at 3. A occupies the angular space between 3 5 and 3 4, 3 4 and 4 5, and 4 5 and 5 3, and so on.

4th. Draw a line of loads (vertical in this case) representing the whole load W, and divide it into parts representing the external forces acting at the joints.

As the supporting forces here are vertical and equal, they will each be represented by half this line of loads (K P and L P, Fig. 2).

5th. Letter the line of loads to correspond with the lettering of the truss. In reading from K to a, Fig. 1, we cross the line representing the load $\frac{W}{12}$ at 3. Therefore K a on the line of loads, must be $\frac{W}{12}$. For a similar reason $a\ b = \frac{W}{6}$, and so on.

Since the loads at 2 and 3 rest directly and vertically upon the supports of the truss (and the supporting forces are vertical), they produce no stress on the frame, and may be left out of account.

Then $a\ h$ upon the line of loads will represent the effective load, and a P and h P the supporting forces to sustain that load. The loads at 2 and 3, and the letters K and L could, therefore, be omitted in Figs. 1 and 2.

6th. Commence at some joint where all but two of the forces are known, draw the polygon of forces for that joint, and by proceeding from one joint to another complete the diagram.

Commencing at the left hand joint of the truss Fig. 1, and leaving out the load $\frac{W}{12}$ as above directed, we have on the line of loads, Fig. 2, the distance a P, upon which to construct a triangle having its sides parallel to 3 4 and 3 5.

Draw through a, Fig. 2, a line parallel to 3 4, Fig. 1, and through P a line parallel to 3 5, and mark their intersection by the letter A.

Then pass to the joint at 4, draw lines through A and b of Fig. 2, parallel to 4 5 and 4 6 of Fig. 1, and mark their intersection with B. Through B draw a line parallel to 5 6 to meet a line through P, parallel to 5 7, and mark the intersection C. In a similar manner complete the diagram represented by Fig. 2.

The advantage of this notation lies in the fact that, if the letters in the angular spaces about any joint are taken in order, and the same letters, taken in the same order, are found in the diagram, at the vertices of the polygon representing the forces at that joint, the diagram is correct. Thus in Fig. 1, for the joint 3, we have aAPa, and in Fig. 2, aAPa forms a closed polygon. Reading about joint 1 of the frame we have dDEed, and dDEed forms a polygon in the diagram.

Failing Cases.—Whenever we find on arriving at any joint,

more than two unknown forces, this method apparently fails, since an indefinite number of polygons may then be drawn with corresponding parallel sides. A careful inspection of the truss in connection with a sketch of the general outline of the diagram will often reveal some condition fixing the relations of the lines in the diagram.

This is the case in Fig. 77, A. M. Arriving at joint 5, we find the forces parallel to 5 4, 5 7, and 5 1 unknown ; but by sketching a diagram for this joint, we see that the lines representing the stresses on 4 7 and 1 7 must overlap or be portions of the same

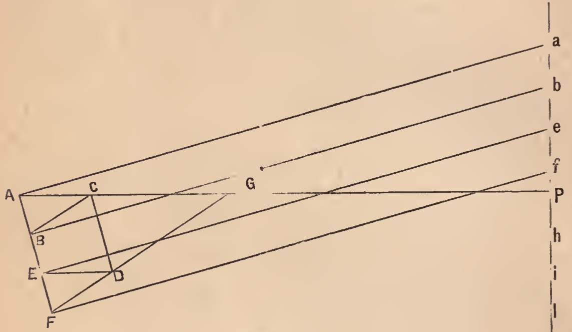


Fig. 3.

Diagram of stresses for half the truss represented by Fig. 77, A. M. Letter that figure in a manner illustrated by Fig. 1.

line, in order to read correctly for the joints 4 and 7 ; and since 5 7 and 7 1 make equal angles with 1 3, the lines on the diagram representing their stresses must intersect midway between lines drawn through *e* and *f*, parallel to 1 3. Hence, as CD is the line to be drawn, we may draw the line from C, parallel to 5 4, to a point D, half way between the parallel lines above mentioned.

There will be no difficulty with the remaining part of the problem.

Or otherwise : Since the stresses on 5 7 and 5 6 are seen to be equal, and make equal angles with the rafter, we may draw an auxiliary line from B parallel to 5 4, till it meets a line through

NOTE.—For an extended discussion and application of these principles, the student is referred to "Economics of Construction," by R. H. Bow, C. E.

e parallel to the rafter, which gives E. From E and C draw lines parallel to 5 7 and 5 4, which will intersect in D. It will be seen that this makes ED = BC, or the stress upon 5 7 equal to the stress upon 5 6.

PROBLEMS.—Draw diagrams for figures 79 and 82, A. M.

ARTICLE 160.

EXAMPLE 1. The T_{67} of the first, and T_{45} of the last equations at the end of this example, should be omitted.

ARTICLE 167.

The funicular polygon is defined in Article 152 instead of Article 150.

ARTICLE 169.

Equation 8.—Combine the equation $x_1 + x_2 = a$, with the equation obtained from the proportion $y_1 : y_2 :: x_1^2 : x_2^2$.

Equation 9. See equation 5; $m = \frac{x_1^2}{4y_1} = \frac{x_2^2}{4y_2}$; $x_1^2 = 4\,m\,y_1$, $x_2^2 = 4\,m\,y_2 \ \therefore x_1^2 + x_2^2 = 4\,m\,(y_1 + y_2)$, $\therefore m = \frac{x_1^2 + x_2^2}{4\,(y_1 + y_2)}$.

The remaining part of the equation can be obtained by substituting the values of x_1 and x_2 in equation 8.

Equation 16. $s = \int ds = \int \sqrt{d\,x^2 + d\,y^2} = \int d\,y \sqrt{1 + \frac{d\,x^2}{d\,y^2}}$.

From the equation of the curve, $x^2 = 4\,m\,y$; $\frac{dx}{dy} = \frac{2\,m}{x} = \sqrt{\frac{m}{y}}$,

$\therefore s = \int d\,y \sqrt{\frac{m+y}{y}} = 2 \int d\,z \sqrt{m + z^2}$ in which $z^2 = y$.

Then by Integral C, we have

$$s = 2 \left\{ \frac{z}{2} \sqrt{m + z^2} + \frac{m}{2} \log_e \left(z + \sqrt{m + z^2}\right) + C \right\}.$$

$$= \sqrt{y} \sqrt{m + y} + m \log_e \left(\sqrt{y} + \sqrt{m + y}\right) + C.$$

When $y = 0$, $s = 0$. $\therefore 0 = m \log_e \sqrt{m} + C$ or $C = -m \log_e \sqrt{m}$,

$$\therefore s = \sqrt{y(m+y)} + m\left(\log_e\left(\sqrt{y} + \sqrt{m+y}\right) - \log_e \sqrt{m}\right).$$

$$= \sqrt{y^2 + m.y} + m \log_e \left(\frac{\sqrt{y} + \sqrt{m+y}}{\sqrt{m}}\right).$$

$$= \sqrt{y^2 + \frac{x^2}{4}} + \frac{x^2}{4\,y} \log_e \left(\frac{y + \sqrt{y^2 + \frac{x^2}{4}}}{\frac{x}{2}}\right).$$

$\sqrt{y^2 + \frac{x^2}{4}} = \frac{x}{2}\sqrt{\frac{4\,y^2}{x^2} + 1} = m \tan i \,.\, \sec i.$ See equation 6.

$$\therefore s = m \left\{ \tan i \,.\, \sec i \,. + \log_e (\tan i + \sec i) \right\}.$$

To obtain equation 17 develop $\left(\frac{m}{y} + 1\right)^{\frac{1}{2}}$. by the binomial formula, multiply by dy and integrate two terms of the series.

Article 171.

The last of equations 1. Place the letter G at the intersection of AB and EF prolonged.

Then $DE = \frac{1}{2}\ EF = \frac{1}{2}\ EG \,.\, \frac{\frac{W}{2}}{P + \frac{W}{2}} \Big($since $EF : EG :: ef : eb :: \frac{W}{2} : P + \frac{W}{2}\Big) = \frac{1}{8}\ BC \,.\, \frac{W}{P + \frac{W}{2}}$

(since $E\,G = \frac{1}{2}\ B\,C$, for $A\,E = \frac{1}{2}\ A\,C$).

At the top of page 171.

Place the letter A in its proper position in Fig. 86. y_1 is the projection of the line DE upon a line drawn perpendicular to AC.

Article 172.

Equation 11. In Fig. 87, A. M., draw lines through B and x perpendicular to the arrow line through P. It will be seen that $t \,.\, \cos i = \frac{x}{2} \,.\, \cos j \;\therefore\; i = \text{arc} \,.\, \cos \left(\frac{x}{2\,t} \cos j\right)$.

Equation 12. Equation 16 of Art. 169, gives the length of a parabola, from its vertex to a point whose tangent makes the angle i with the axis of x, tangent at the vertex. Equation 12 gives the length of that portion of the curve included between two points where the inclinations are i and j. This equation is obtained directly from equation 16 Art. 169, by taking that integral between the limits i and j.

Equation 13.—In equation 17, Art. 169, substitute $x + y \sin j$ for x, and $y \cos j$ for y. These values are obtained by an inspection of Fig. 87, A. M.

ARTICLE 174.

Equation (a). $\frac{d^2 u}{d x^2} = \frac{u}{a^2}$; $\frac{2\, du\ d^2 u}{dx^2} = \frac{2\, u\, d u}{a^2}$ or $d \left(\frac{d u^2}{d x^2}\right)$ $= d \left(\frac{u^2}{a^2}\right)$. $\therefore \frac{d u^2}{d x^2} = \frac{u^2}{a^2} + c$, $\therefore dx = \frac{a\ d\ u}{\sqrt{u^2 + a^2 c}}$.

Let $\sqrt{u^2 + a^2 c} = z + u$
then $u^2 + a^2 c = z^2 + 2\, z\, u + u^2$ or $a^2 c = z^2 + 2\, z\, u$. (A)
Differentiate and divide by 2.

$$0 = z\, d\, z + z\, d\, u + u\, d\, z \text{ or } d\, u = -\frac{z + u}{z} \cdot d\, z$$

$\therefore d\, x = -a \frac{d\, z}{z}$ $\therefore x = -a \log_e z + a \log_e c_1$. In which $a \log_e c_1$ is the constant of integration.

$$\therefore \frac{x}{a} = -\log_e z + \log_e c_1 = \log_e \frac{c_1}{z}.$$

then $e^{\frac{x}{a}} = \frac{c_1}{z} \therefore z = \frac{c_1}{e^{\frac{x}{a}}} = c_1 e^{\frac{-x}{a}}$

From equation A above, we have $u = \frac{a^2 c - z^2}{2\, z}$

$$\therefore u = \frac{a^2 c - c_1^2 e^{\frac{-2x}{a}}}{2\, c_1 e^{\frac{-x}{a}}} = \frac{a^2 c}{2\, c_1} e^{\frac{x}{a}} - \frac{c_1}{2} e^{\frac{-x}{a}} = \text{A}\, e^{\frac{x}{a}} - \text{B}\, e^{\frac{-x}{a}}$$

Equation (*c*).—From Equation 1 we have $\frac{d\, u}{d\, x} = y$. Hence, differentiate equation (*b*), and divide by $d\, x$; the result will be equation (*c*).

Equation 6 is derived from (4), by multiplying by $2\ e^{\frac{x}{a}}$, which reduces it to the form

$$2\ \frac{y}{y_0}\ e^{\frac{x}{a}} = e^{\frac{2x}{a}} + 1\ ;\ e^{\frac{2x}{a}} - 2\frac{y}{y_0}\ e^{\frac{x}{a}} = -1$$

$$\therefore\ e^{\frac{x}{a}} = \frac{y}{y_0} \pm \sqrt{\frac{y^2}{y_0^2} - 1}\ ;\ \frac{x}{a} = \log_e \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1}\right)$$

$$\therefore\ x = a \log_e \left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1}\right).$$

ARTICLE 175.

Equation 4.—By squaring both members of equation 3, and solving with respect to $\frac{dx}{ds}$ we find $\frac{dx}{ds} = \frac{m}{\sqrt{m^2 + s^2}}$. The numerator of the second member is m, and not m^2.

Equation 5.—Integrate equation 4 by means of integral B.

Equation 8.—Divide both members of Equation 5 by m, and write the resulting equation in the form $e^{\frac{x}{m}} = \frac{s}{m} + \sqrt{1 + \frac{s^2}{m^2}}$. Substitute the value of $\frac{s}{m}$ from Equation 6, for the first term of the last member, and transpose. The result is $\frac{1}{2}\left(e^{\frac{x}{m}} + e^{\frac{-x}{m}}\right) = \sqrt{1 + \frac{s^2}{m^2}}$. This value in the first part of the first value of y (Equation 8), gives $y = \sqrt{s^2 + m^2} - m$.

Equations 11.—Combine $x_1 + x_2 = h$, and $x_1 - x_2 = k$.

Equations 12.—Putting these values of x in Equations 6 and 8, the following will be found to be the correct results:

$$l = s_1 - s_2 = \frac{m}{2}\left(e^{\frac{h}{2m}} + e^{\frac{-h}{2m}}\right).\left(e^{\frac{k}{2m}} - e^{\frac{-k}{2m}}\right).$$

$$v = y_1 - y_2 = \frac{m}{2}\left(e^{\frac{h}{2m}} - e^{\frac{-h}{2m}}\right).\left(e^{\frac{k}{2m}} - e^{\frac{-k}{2m}}\right).$$

NOTE.—The reference in the last line of page 173, should be Article 174, instead of 173, as in some editions.

Properties of the Catenary.

I. The radius of curvature may be found by making proper substitutions in the formula

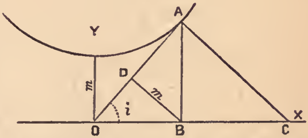
$$\rho = \frac{\left(1 + \frac{d y^2}{d x^2}\right)^{\frac{3}{2}}}{\frac{d^2 y}{dx^2}}; \text{ or as follows:}$$

$$\rho = \frac{d s^3}{d x\, d^2 y} \text{ in which } d s = (d x^2 + d y^2)^{\frac{1}{2}}.$$

From Equations 1 and 3, $\frac{d\ y}{d\ x} = \frac{s}{m} \therefore \frac{d^2 y}{d x} = \frac{d s}{m}$ or

$$d^2 y = \frac{d x.\ d s}{m} \therefore \rho = \frac{d s^3}{\frac{d x . d x . d s}{m}} = m \left(\frac{d s}{d x}\right)^2 = m \cdot \sec^2 i.$$

Fig. 4. *



II. From the Equation $y^2 = s^2 + m^2$,(obtained from 8 by transferring the origin of coördinates to a distance m below the vertex,)it will be seen that, if a right triangle be constructed, having y for its hypotenuse, and m for the side adjacent to the foot of the ordinate, the remaining side will be s, or the length of the curve from the vertex to the point having y as its ordinate.

From Equations 1 and 2, we have $\tan i = \frac{d y}{d x} = \frac{s}{m}$.

Hence, if the lines are drawn as B D and D A in Fig. 4, $\tan i$ $= \tan \angle \text{A B D} = \frac{\text{A D}}{\text{D B}} = \frac{s}{m}$; hence, the line A D is tangent to the curve at A.

If we draw A C perpendicular to A D, we have from the similar triangles A B D and A B C, $m : y :: y :$ A C $\therefore$ A C $= \frac{y^2}{m} = m\left(\frac{y}{m}\right)^2 = m \cdot \sec^2 \text{A B D} = m \cdot \sec^2 i = \rho$. See Equation 17.

† NOTE.—The line A D, prolonged, does not necessarily pass through O.

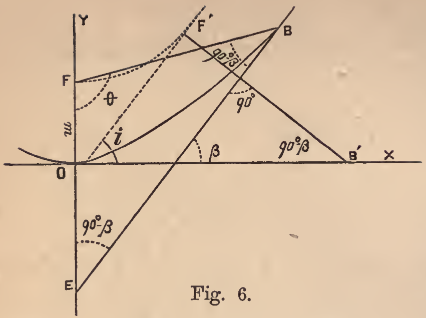


Fig. 6.

IV. In any position of the rolling parabola, a line drawn from the focus to the point of contact, will be normal to the curve described by the focus, since it is the instantaneous radius vector of the describing point. (See A. M. at the bottom of page 401.)

The following shows that this line is equal to $m \cdot \sec^2 i$:

Let B, in Fig. 6, be any point in a parabola whose focal distance is m, F its focus, and B E a tangent at the point B, making the angle β with the axis O X. The angle $E = < B$ (from Analytical Geometry), $= 90^{\theta} - \beta$, from figure.

When the parabola has rolled till B comes into O X at B^1, the line B E will coincide with O X, the line F B will be at F^1 B^1 and make an angle $= 90^{\theta} - \beta$ with O X, since a line drawn from B to the focus F will retain its position relative to the tangent, and a line drawn through $F^1 \perp F^1$ B^1 will be tangent to the curve described by the focus, since, as before stated, F^1 B^1 is normal to that curve.

Then $\beta = i$.

From the polar equation of the parabola, we have $\rho = \dfrac{2\,m}{1 + \cos\theta}$ when $\theta = < B\ F\ E$. (See Church's Analytical Geometry, pp. 128 and 123.)

$\theta = 180° - 2\,(90° - \beta) = 2\,\beta = 2\,i$

$\cos\theta = \cos 2\,i = 1 - 2\sin^2 i$

$$\therefore\ \rho = \frac{2\,m}{2\,(1 - \sin^2 i)} = \frac{m}{\cos^2 i} = m.\ \sec^2 i = F\,B = F^1\,B^1.$$

Therefore as F^1 $B^1 = m \cdot \sec^2 i$, and is normal to the curve described, it follows by the first and second geometrical properties that this curve is a catenary.

ARTICLE 181.

The Geometrical Construction preceding Equations 5.

At this point we need to show :

1st. That $2\ \overline{O'm} = A + B$.
2d. That ap and aq are equal to A and B.
3d. That by construction, the triangle $p\,O'q$ has a right angle at O′.
4th. That $q\,p$ is perpendicular to $O'a$, and tangent to the ellipse.
5th. That the angle $p\,O'a$ is that made by a perpendicular to the tangent qp, with the major axis of the ellipse.

1st.

Complete a parallelogram upon $O'a$ and $O'B'$, and call the vertex opposite O′, E. Then by construction, the point E is in the prolongation of $O'm$, which is a diagonal of this parallelogram.

From Trigonometry, $\overline{O'E}^2 = (2\ O'm)^2 = \overline{O'B'}^2 + \overline{O'a}^2 + 2\ O'B' . O'a . \cos B'O'a$.

By an examination of Fig. 89, A. M., it will be seen that $\angle B'O'a = \angle j$. Since $O'A'$ and $O'B'$ are conjugate diameters of the ellipse, and $O'a = O'A'$ by construction, let $O'B' = A'$ and $O'a = B'$.

Then we have from the preceding equation, $(2\ \overline{O'm})^2 = A'^2 + B'^2 + 2\ A'B' \cos j$.

$\cos j = \cos . (90° - A'O'B') =$ sin of the angle between the conjugate diameters A′ and $B' = \frac{A\ B}{A'\ B'}$; and $A'^2 + B'^2 = A^2 + B^2$. See equations 2 and 3 of Art. 157, Church's Analytical Geometry.

$$\therefore (2\ O'm)^2 = A^2 + B^2 + 2\ AB \text{ or } 2\ O'm = A + B \Big\}.$$

2d.

In a similar manner we obtain $B'a = 2\ ma = A - B$, $\therefore ma = \frac{A - B}{2}$.

Since $mp = mq = O'm = \frac{A + B}{2}$,

$$\left.\begin{aligned} ap &= mp + ma = \frac{A + B}{2} + \frac{A - B}{2} = A, \\ aq &= mq - ma = \frac{A + B}{2} - \frac{A - B}{2} = B, \end{aligned}\right\}$$

3d.

From Geometry $2\,\overline{mp}^2 + 2\,\overline{O'm}^2 = \overline{O'p}^2 + \overline{O'q}^2 = (2\,O'm)^2 = (A + B)^2 = \overline{pq}^2$,

$\therefore$ as the square of pq is equal to the sum of the squares of $O'q$ and $O'p$, the triangle has a right angle at O'.

4th.

In the equation on page 187, A. M., $c = \sec j = \dfrac{1}{\cos j}$ as will be seen by an inspection of Fig. 89, A. M. By construction, $\dfrac{O'a}{O'B'} = \dfrac{r}{c\,r} = \dfrac{1}{c} = \cos j = \cos B'O'a$. This last value is obtained from the figure, where it will be seen that $j = B'O'a$.

Then as $\cos B'O'a = \dfrac{O'a}{O'B'}$, it follows that the angle formed by joining $B'a$, is 90° at a, also that as the angle at a is 90°, the line $q\,p$ is parallel to $A'O'$, and therefore tangent to the ellipse at B', the extremity of the diameter conjugate to $O'A'$.

5th.

Call the angle $p\,O'a$, B'. Draw perpendiculars from a to $O'p$ and $O'q$, and call their intersections with these lines, E and F.

Then $aE = ap \sin \angle p = A \sin (90 - B') = A \cos B'$.

$aF = aq \sin \angle q = B \sin B'$.

$\overline{O'a}^2 = \overline{aE}^2 + \overline{aF}^2 = A^2 \cos^2 B' + B^2 \sin^2 B'$.

By comparing this equation with equation 13, Art. 95, A M., and the notes upon that equation, it will be seen that $B' = \angle p\,O'a =$ the angle that a normal to a tangent makes with the major axis of the ellipse. Hence $O'p$ is the direction of the major axis, and $O'q$ at right angles with it, the direction of the minor axis.

Equations 6 are obtained from the equation of the length of any diameter of an ellipse, in terms of the axes and the angle made by the diameter with one of the axes.

Let k be the angle made with the major axis, and k' the angle made with the minor axes.

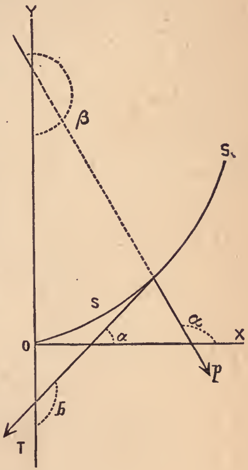
Then $c^2 r^2 = \sqrt{\dfrac{A^2 B^2}{A^2 \sin^2 k + B^2 \cos^2 k}} = \sqrt{\dfrac{A^2 B^2}{A^2 \cos^2 k' + B^2 \sin^2 k'}}$.

(Church's Analytical Geometry, page 169.)

The last value in equation 6, which is the value of sin k' obtained by solving the above equation, should be

$$\frac{A}{c\ r}\sqrt{\frac{c^2\ r^2 - B^2}{A^2 - B^2}}.$$

Article 182.



Theorem I.—Let O s_1 represent any flexible and inextensible cord, fixed at s_1 and O in the plane X Y, and acted upon by forces in that plane.

Let p be the force per unit of length of the curve, and α and β the angles that p makes with the axes OX and OY.

Let T = tension at any point, making the angles a and b with OX, OY. Let T_0 = tension at O, making angles a_0 and b_0 with OX, and OY. Let s = any length of the curve measured from O. Let s_1 = whole length of the curve measured from O.

Resolving all the forces parallel to the axes $\left(\text{since } \cos a = \frac{d\ x}{d\ s}\right.$ and $\left.\cos b = \frac{d\ y}{d\ s}\right)$, we have

$$\mathrm{T}\frac{d\ x}{d\ s} + \int_s^{s_1} p\ .\ ds\ .\ \cos\alpha - \mathrm{T}_0 \cos a_0 = 0. \qquad (1)$$

$$\mathrm{T}\frac{d\ y}{d\ s} + \int_s^{s_1} p\ .\ ds\ .\ \cos\beta - \mathrm{T}_0 \cos b_0 = 0. \qquad (2)$$

Differentiate equations 1 and 2.

$$\mathrm{T}\ .\ d\left(\frac{d\ x}{d\ s}\right) + \frac{d\ x}{d\ s}\ .\ d\,\mathrm{T} = -\,p\ .\ ds\ .\ \cos\alpha. \qquad (3)$$

$$\mathrm{T}\ .\ d\left(\frac{d\ y}{d\ s}\right) + \frac{d\ y}{d\ s}\ .\ d\,\mathrm{T} = -\,p\ .\ ds\ .\ \cos\beta. \qquad (4)$$

Multiply 3 by $\frac{dx}{ds}$, and 4 by $\frac{dy}{ds}$, and add.

$$T\left(\frac{dx}{ds}\cdot d\left(\frac{dx}{ds}\right)+\frac{dy}{ds}\cdot d\left(\frac{dy}{ds}\right)\right)+d\,T\left(\frac{dx^2+dy^2}{ds^2}\right)=-p\,ds$$

$$\left(\frac{dx}{ds}\cos\alpha+\frac{dy}{ds}\cos\beta\right)\cdot \qquad \text{But } \frac{dx^2+dy^2}{ds^2}=\frac{ds^2}{ds^2}=1.$$

$$\text{And } \frac{dx}{ds}\cdot d\left(\frac{dx}{ds}\right)+\frac{dy}{ds}\cdot d\left(\frac{dy}{ds}\right)=\tfrac{1}{2}\,d\left(\frac{dx^2+dy^2}{ds^2}\right)=0.$$

Then $d\,T = -p\,(\cos a\,.\cos\alpha + \cos b\,.\cos\beta)\,ds = -p\,.\,ds\,.$ $\cos\theta$. (5)

In which $\cos.\theta =$ the cosine of the angle between the direction of the force p, and a tangent to the curve. For in general, $\cos\theta = \cos(\alpha - a) = \cos\alpha\cos a + \sin\alpha\sin a = \cos\alpha\cos a + \cos\beta\,.\cos b$.

If the force p is normal to the curve, then $\theta = 90^\circ$, $\cos\theta = 0$, and T will be constant, since its differential is 0.

In this case, equations 3 and 4 reduce to

$$T\,.\,d\left(\frac{dx}{ds}\right) = -p\,.\,ds\,.\cos\alpha\,;\quad T\,.\,d\left(\frac{dy}{ds}\right) = -p\,.\,ds\,.\cos\beta.$$

Square and add the above equations.

$$T^2\left[\left(d\,.\frac{dx}{ds}\right)^2+\left(d\frac{dy}{ds}\right)^2\right]=p^2\,ds^2\,(\cos^2\alpha+\cos^2\beta)=p^2\,ds^2.$$

$$\therefore T^2 = \frac{p^2\,ds^4}{(d^2x)^2+(d^2y)^2}.$$

$$T = p\,.\,\frac{ds^2}{\sqrt{(d^2x)^2+(d^2y)^2}} = p\,\rho. \tag{6}$$

In which ρ is the radius of curvature. (See Church's Calculus, Art. 102.)

Equations 2.—See Church's Calculus, Art. 106.

Article 183.

The equation above 6 should be $w\int_{x_0}^{x_1} x\,dx = w\,\frac{x_1^2 - x_0^2}{2}$, and the equation above 8 should be $w\int_{x}^{x_1} x\,dx = w\,\frac{x_1^2 - x^2}{2}$.

Article 185.

The equation following 12. When $i = 0$, the fraction $\frac{i - \cos\,.\,i\sin i}{2\sin^3 i}$ is indeterminate. Its value is found by

taking the third differential co-efficient of the numerator and denominator, and introducing the value $i = 0$. The result is $-\frac{1}{3}\cdot$ Hence for $i = 0$, $p_y = w\,r\left(m - \frac{1}{3}\right)\cdot$

Equation 13 is found in a form more available for practical use, on page 423 of Rankine's Civil Engineering.

Equation 16.—The $\cos j$ of this equation should be $\sin j$.

Article 187.

Equation 4.—In Equation 2 we have $H_0 =$ max. value of $P_x \cdot \frac{d\,y}{d\,x}$ max. value of $= \frac{P_x}{\frac{d\,x}{d\,y}}\cdot$ Differentiating both numerator and denominator with respect to y to find the value of this vanishing fraction, we have $H_0 = \frac{\frac{d\,.\,P_x}{d\,y}}{\frac{d\,.\,d\,x}{d\,y\,.\,d\,y}}$ (when $y = 0$). But for $y = 0$, $\frac{d\,P_x}{d\,y} = p_0$. Hence, $H_0 = \frac{p_0}{\frac{d^2\,x}{d\,y^2}}$ (for $y = 0$).

Article 191.

Equation 2. $P \sin\phi = \frac{P}{\operatorname{cosec}\phi} = \frac{P}{\sqrt{1 + \cot^2\phi}} =$

$$\frac{P}{\sqrt{1 + \frac{1}{\tan^2\phi}}} = \frac{P}{\sqrt{1 + \frac{1}{f^2}}} = \frac{f\,P}{\sqrt{1 + f^2}}.$$

Article 197.

Equations 5 and 6.—The last term of the denominator of the last fraction of each equation should be $\cos^2\phi$.

Equation 9.—In the equation $\cos 2\,\psi = \frac{2\ p_x \cos\theta - p_1 - p_2}{p_1 - p_2}$, substitute the values of p_x, p_1, and p_2, from Equations 1, 7, and 8, and reduce to the form $\cos 2\,\psi = \frac{\cos^2\theta - 1}{\sin\phi} +$

$$\frac{\cos\theta\ \sqrt{\cos^2\theta - \cos^2\phi}}{\sin\phi}. \quad \text{But } \frac{\cos^2\theta - 1}{\sin\phi} = -\left(\frac{1-\cos^2\theta}{\sin\phi}\right)$$

$$= -\sin\theta\,\frac{\sin\theta}{\sin\phi}.$$

$$\frac{\sqrt{\cos^2\theta - \cos^2\phi}}{\sin\phi} = \sqrt{\frac{(1-\cos^2\phi)-(1-\cos^2\theta)}{\sin^2\phi}} = \sqrt{1-\frac{\sin^2\theta}{\sin^2\phi}}$$

$$\therefore \cos 2\psi = \cos\theta\sqrt{1-\frac{\sin^2\theta}{\sin^2\phi}} - \sin\theta\,\frac{\sin\theta}{\sin\phi}$$

$$= \cos\theta\,.\,\cos\sin^{-1}\frac{\sin\theta}{\sin\phi} - \sin\theta\,.\,\sin,\sin^{-1}\frac{\sin\theta}{\sin\phi}$$

$$= \cos\left(\theta + \sin^{-1}\frac{\sin\theta}{\sin\phi}\right) \therefore \psi = \frac{1}{2}\left(\theta + \sin^{-1}\frac{\sin\theta}{\sin\phi}\right).$$

By changing the construction of Fig. 57, A. M., as directed in Case 2 of Art. 112, to adapt it to this case, it will be evident that $2\,\widehat{x\,n'} = 2\,\psi$, as measured in that figure, is $> 180°\ \therefore\ \psi$ is $> 90°$. Therefore $\sin^{-1}\frac{\sin\theta}{\sin\phi}$ must be greater than 90°, in order that Equation 9 may be true.

The reference near the bottom of page 216, should be to Problem IV, &c.

Article 198.

Problem.—Find the pressure against a vertical wall 12 feet high, sustaining a bank of earth which slopes backward from the top of the wall at an angle of 30°, the angle of repose of the earth being 45°, and its weight 100 lbs. per cubic foot.

Ans. $3600\left(\frac{3-\sqrt{3}}{\sqrt{3}+1}\right) = 1671$ lbs. nearly. (Length of wall $= 1$ foot.)

Article 199.

Problem.—A prismatic column of solid masonry 80 feet high, and weighing 120 pounds per cubic foot, is to be built upon a foundation of earth whose angle of repose is 30°. What is the least depth below the surface of the ground at which the foundation course should rest, if the surface of the ground is horizontal, and the earth weighs 100 lbs. per cubic foot?

Ans. $10\frac{2}{3}$ ft.

Equation 5.—Draw a trapezoid whose base b is horizontal, and whose parallel sides are vertical and equal to p' and $w\,x$, respectively. The ordinates of this trapezoid will represent intensity of pressure at any point, and its area, the total pressure upon the surface. Divide the trapezoid into a triangle and a rectangle by a line through one extremity of the shorter vertical, parallel to the base. The distance $c\ b$ is obtained by equating the moment of the whole trapezoid about the center of b, to the sum of the moments of the above mentioned rectangle and triangle. The moment of the rectangle $= 0 \therefore b \cdot \frac{p' + w\,x}{2} \cdot b\,c$

$$= \frac{p' - w\,x}{2} \cdot b \cdot \frac{b}{6}.$$

$c = \frac{p' - w\,x}{6\,(p' + w\,x)}$. In this equation introduce the value of p' as given in Equation 2 of this article, and the resulting equation will be the last value of c given in Equation 5.

Article 202.

Equations 10 *and* 11.—From what precedes these equations, it would seem that they should be obtained by substituting $\frac{s}{s+1}$ and $\frac{1}{s+1}$ for $\frac{1}{2}$ in Equations 5 and 7 respectively. The following are the results obtained by such a substitution:

$\frac{s}{2\,(s+1)}\,w\,t\,(x^2 + h\,x)$ and $\frac{1}{s+1}\,w\,l\,(x^2 + h\,x)$.

Article 214.

To obtain Equations 4 and 5 from Equations 1 and 2, divide the numerators and denominators of the second members of those equations by $\sqrt{x}$, and then make $x =$ infinity in the resulting equations.

Article 215.

Equation 7.—Conceive of a circular chimney whose external and internal diameters are t and $t - 2\,B$ respectively. The outer circumference is $\pi\,t$, which, multiplied by b, and placed equal to the sectional area of the masonry, gives

$$\pi\,b\,t = \frac{\pi\,t^2}{4} - \frac{\pi}{4}\,(t - 2\,B)^2 \therefore b = B\left(1 - \frac{B}{t}\right).$$

ARTICLE 217.

NOTE.—The arm of the couple will be easily obtained by drawing from D and F, Fig. 101, A. M., lines perpendicular to H P; from F a line parallel to H P, and a horizontal line through D. The first of these lines is $\frac{x}{3} \cos \theta$. The remaining term $(q + \frac{1}{2}) t . \sin (\theta + i)$ is easily obtained.

ARTICLE 225.

"Find the center of gravity of the load between the joint of rupture C, and the crown A, and draw through that center of gravity a vertical line." See Fig. 107, A. M. "Then if it be possible, from one point in that vertical line, to draw a pair of lines, one parallel to a tangent to the soffit at the joint of rupture, and the other parallel to a tangent to the soffit at the crown, so that the former of these lines shall cut the joint of rupture, and the latter the keystone, in a pair of points which are both within the middle third of the arch ring, the stability of the arch will be secure; and if the first point be the point of rupture, the second will be the center of resistance at the crown of the arch, and the crown of the true line of pressures." Rankine's Civil Engineering, page 442. Let the student make the above calculation and construction for an assumed circular arch.

ARTICLE 234.

Example 1.—The value of P_x is obtained by remembering that the thickness is small compared with the radius of curvature, and that the surface of a zone is equal to the circumference of a great circle, multiplied by the altitude of the zone.

ARTICLE 235.

Equation 5.—This equation will be found in Chauvenet's Trigonometry, Formula 53, Page 163, and Equation 6 in Formula 355, Page 256.

ARTICLE 249.

See reference made in Article 263, to this Art.

ARTICLE 260.

After $\overline{CD}$ in equations 1, read $a x$. At the end of next to the last line on page 282, read, principal elementary *strains.*

ARTICLE 271.

Equation 1.—See Art. 179, Theorem.

ARTICLE 273.

Equation 9.—In Equation 6, let $p_0 = f$, divide both numerator and denominator of the last value of p_0 by r^2, and solve the resulting equation with reference to $\frac{R}{r}$.

Figure 119.—The general equation of Hyperbolas of Higher Orders, is $y^m x^n = a$.

$x y = a$ is the equation of a hyperbola of the first order, referred to its asymptotes, $x y^2 = a$ is the equation of a hyperbola of the second order.

$\therefore x y^2 = a = x' y'^2.$ (A)

Take O R as the axis of Y, and a vertical through O, as the axis of X.

Then we have

$x : x' :: y'^2 : y^2$ from equation (A)

$$\left.\begin{aligned} \overline{rA} : \overline{RB} &:: R^2 : r^2 \\ &:: \overline{OR}^2 : \overline{Or}^2 \end{aligned}\right\} \text{See A. M.}$$

By comparing these proportions with equation A, we have

$\overline{rA} \times r^2 = \overline{RB} \times R^2 = a.$ (B)

The area $r\,A\,B\,R = \int_o^x \int_r^R dx \,.\, dy = \int_r^R x\, dy = \int_r^R \frac{a}{y^2} dy$

$$= a\left(\frac{-1}{y}\right)_r^R = a\left(\frac{1}{r} - \frac{1}{R}\right). \tag{C}$$

$$= \frac{a}{r} - \frac{a}{R} = \frac{\overline{rA} \times r^2}{r} - \frac{\overline{RB} \times R^2}{R} = \overline{rA} \times r - \overline{RB} \times R. \tag{D}$$

By comparing equation C with the last equation in case 2, it will be seen that this area represents case 2, and that $a = \overline{rA} \times r^2$. (E)

From the proportion $\overline{CA} : \overline{DB} :: q_0 : q_1$, we have $\overline{CA}\, q_1 = \overline{DB}\, q_0$. By introducing the values of q_1 and q_0, as found in the equations at the bottom of page 292, A. M., the equation

becomes $\left(\frac{a}{R^2} - m\right) \overline{CA} = \left(\frac{a}{r^2} - m\right) \overline{DB} = (\overline{RB} - m)\, CA$
$= (\overline{rA} - m)\ \overline{DB}$

$$\overline{RB} \times \overline{CA} - \overline{rA} \times \overline{DB} = m\,(\overline{CA} - \overline{DB}) = m\,(\overline{rA} - \overline{RB})$$
$$\overline{RB}\,(\overline{rA} - \overline{rC}) - (\overline{rA}\,.\,\overline{RB} - \overline{rC}) = m\,(\overline{rA} - \overline{RB})$$
$$\overline{rC}\,(\overline{rA} - \overline{RB}) = m\,(\overline{rA} - \overline{RB})$$
$$\therefore m = \overline{rC}.$$

Hence it will be seen that $\overline{rC}$ represents m of case 1, in the solution of that case.

Since $\overline{rC}$ is m, and $\overline{rA} \times r^2 = a$,

We have from equation 4, $q = \frac{a}{r'^2} - m = \frac{\overline{rA} \times r^2}{r'^2} - \overline{rC}.$

But $\overline{rA} : x' :: r'^2 : r^2 \therefore \frac{rA \times r^2}{r'^2} = x'.$

$\therefore q = x' - \overline{rC} =$ the segment of the ordinate corresponding to $y = r'$, between $\overline{CD}$ and the curve A B.

Again, $p = \frac{a}{r'^2} + m = x' + \overline{rC} = x' + \overline{rE}.$

$=$ the entire ordinate from E F to A B.

From 6, we have

$$p_0 = \frac{a}{r^2} + m = \frac{rA \times r^2}{r^2} + \overline{rC} = \overline{rA} + \overline{rE} = \overline{AE}.$$

Article 277.

Differentiate Equation 1.

$w\, s\, dx = f\, d\, s$; $\frac{w}{f}\, dx = \frac{d\, s}{s}$. Then $\frac{w\, x}{f} = \log_e S + C.$

When $x = 0$, $S = \frac{W}{f}$; see equation (1) A. M.

Then $0 = \log_e \frac{W}{f} + C$ or $C = -\log_e \frac{W}{f}$.

$$\therefore \frac{w\, x}{f} = \log_e S - \log_e \frac{W}{f} = \log_e \frac{fS}{W}.$$

$$\therefore e^{\frac{w\, x}{f}} = \frac{fS}{W} \text{ or } S = \frac{W}{f}\, e^{\frac{w\, x}{f}}.$$

ARTICLE 295.

Table.—Read $m' = \frac{y}{h}$, instead of $\frac{h}{y}$.

ARTICLE 298.

Equation 1. Let G_1, G_2, G_3, and G be the centers of gravity of A_1, A_2, A_3, and A respectively.

Take moments about a horizontal line passing midway between the top of the upper flange, and the bottom of the lower flange.

$$A\bar{x} = A_2\left(\frac{h_1+h_2+h_3}{2} - \frac{h_2}{2}\right) - A_1\left(\frac{h_1+h_2+h_3}{2} - \frac{h_1}{2}\right)$$

$$- A_3\left(h_2 + \frac{h_3}{2} - \frac{h_1+h_2+h_3}{2}\right)$$

$$= A_2\left(\frac{h_1+h_3}{2}\right) - A_1\left(\frac{h_2+h_3}{2}\right) - A_3\left(\frac{h_2-h_1}{2}\right).$$

$$\therefore y_b = \frac{h}{2} - \bar{x} = \frac{h}{2} - \frac{(h_1+h_3)A_2 - (h_2+h_3)A_1 - (h_2-h_1)A_3}{2A}. \quad (1)$$

Equation 2.

$$\overline{G_1G} = \frac{h_1+h_2+h_3}{2} - \frac{h_1}{2} + \bar{x} = \frac{h_2+h_3}{2} + \bar{x}$$

$$= \frac{A_2(h_1+h_2+2h_3) + A_3(h_1+h_3)}{2A}. \quad (A)$$

$$\overline{G_2G} = \frac{h_1+h_2+h_3}{2} - \frac{h_2}{2} - \bar{x} = \frac{h_1+h_3}{2} - \bar{x}$$

$$= \frac{A_1(h_1+h_2+2h_3) + A_3(h_2+h_3)}{2A}. \quad (B)$$

$$\overline{G_3G} = \frac{h_1+h_2+h_3}{2} - \left(h_1 + \frac{h_3}{2}\right) + \bar{x} = \frac{h_2-h_1}{2} + \bar{x}$$

$$= \frac{A_2(h_2+h_3) - A_1(h_1+h_3)}{2A}. \quad (C)$$

The moment of inertia of the section, about its neutral axis will be

$$I = \frac{b_1h_1^3 + b_2h_2^3 + b_3h_3^3}{12} + A_1 \times \overline{G_1G}^2 + A_2 \times \overline{G_2G}^2 + A_3 \times \overline{G_3G}^2.$$

See the end of Art. 95, A. M.

$$\therefore \mathrm{I} = \frac{\mathrm{A}_1 h_2 + \mathrm{A}_2 h_2^2 + \mathrm{A}_3 h_2^3}{12} + \frac{1}{4\,\mathrm{A}_2}[\mathrm{A}_1 \mathrm{A}_2^2 \ (h_1 + h_2 + 2\,h_3)^2$$
$$+ 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_2 + 2\,h_3)\,(h_1 + h_3)$$
$$+ \mathrm{A}_1\,\mathrm{A}_3^2\,(h_1 + h_3)^2, + \mathrm{A}_2\,\mathrm{A}_1^2\,(h_1 + h_2 + 2\,h_3)^2 + 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3$$
$$(h_1 + h_2 + 2\,h_3)\,(h_2 + h_3)$$
$$+ \mathrm{A}_2\,\mathrm{A}_3^2\,(h_2 + h_3)^2, + \mathrm{A}_3\,\mathrm{A}_2^2\,(h_2 + h_3)^2 - 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_2 + h_3)$$
$$(h_1 + h_3) + \mathrm{A}_3\,\mathrm{A}_1^2\,(h_1 + h_3)^2]. \qquad \text{(D)}$$

But $2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_2 + 2\,h_3)\,(h_1 + h_3) = 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_3)^2$

$$+ 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_2 + h_3)\,(h_1 + h_3).$$
$$2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_2 + 2\,h_3)\,(h_2 + h_3)^2 = 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\ \ (h_2 + h_3)^2$$
$$+ 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_2 + h_3)\,(h_1 + h_3).$$
$$\therefore 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,[(h_1 + h_2 + 2\,h_3)\,((h_1 + h_3) + (h_2 + h_3)) - (h_2 + h_3)$$
$$(h_1 + h_3)].$$
$$= 2\,\mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,[(h_1 + h_3)^2 + (h_1 + h_3)\,(h_2 \ \ h_3) + (h_2 + h_3)^2].$$
$$= \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,[(h_1 + h_3)^2 + 2\,(h_1 + h_3)\,(h_1 + h_3) + (h_2 + h_3)^2]$$
$$+ \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_3)^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_2 + h_3)^2$$
$$= \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_2 + 2\,h_3)^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3\,(h_1 + h_3)^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3$$
$$(h_2 + h_3)^2.$$

This reduces equation D to the form

$$\mathrm{I} = \frac{\mathrm{A}_1 h_1^2 + \mathrm{A}_2 h_2^2 + \mathrm{A}_3 h_3^2}{12} + \frac{1}{4\,\mathrm{A}^2}\,[(h_1 + h_2 + 2\,h_3)^2\,(\mathrm{A}_1\,\mathrm{A}_2^2 +$$
$$\mathrm{A}_2\,\mathrm{A}_1^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3) + (h_1 + h_3)^2\,(\mathrm{A}_1\,\mathrm{A}_3^2 + \mathrm{A}_3\,\mathrm{A}_1^2$$
$$+ \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3) + (h_2 + h_3)^2\,(\mathrm{A}_2\,\mathrm{A}_3^2 + \mathrm{A}_3\,\mathrm{A}_2^2 + \mathrm{A}_1$$
$$\mathrm{A}_2\,\mathrm{A}_3)]. \qquad \text{(E)}$$

$$\mathrm{A}_1\,\mathrm{A}_2^2 + \mathrm{A}_2\,\mathrm{A}_1^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3 = (\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3)\,\mathrm{A}_1\,\mathrm{A}_2 = \mathrm{A}\,\mathrm{A}_1\,\mathrm{A}_2.$$
$$\mathrm{A}_1\,\mathrm{A}_3^2 + \mathrm{A}_3\,\mathrm{A}_1^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3 = \qquad \mathrm{A}\,\mathrm{A}_1\,\mathrm{A}_3.$$
$$\mathrm{A}_2\,\mathrm{A}_3^2 + \mathrm{A}_3\,\mathrm{A}_2^2 + \mathrm{A}_1\,\mathrm{A}_2\,\mathrm{A}_3 = \qquad \mathrm{A}\,\mathrm{A}_2\,\mathrm{A}_3.$$

These values reduce equation E to the form

$$\mathrm{I} = \frac{\mathrm{A}_1 h_1^2 + \mathrm{A}_2 h_2^2 + \mathrm{A}_3 h_3^2}{12} + \frac{1}{4\,\mathrm{A}}\,[\mathrm{A}_1\,\mathrm{A}_2\,(h_1 + h_2 + 2\,h_3)^2$$
$$+ \mathrm{A}_1\,\mathrm{A}_3\,(h_1 + h_3)^2 + \mathrm{A}_2\,\mathrm{A}_3\,(h_2 + h_3)^2]. \qquad (2)$$

Equation 3.—See Art. 294, A. M.

Equation 4.—This equation can be obtained by considering h_1, h_2 and A_3 as being so small that they may be left out of consideration. We should then have $\mathrm{A} = \mathrm{A}_1 + \mathrm{A}_2$ and $h' = h_3 = h$. These values in equations 1 and 2, give

$$y_b = \frac{h^1}{2} - \frac{h' A^2 - h' A'}{2 A} = \frac{h^1}{2}\left(\frac{A_1 + A_2 - A_2 + A_1}{A}\right) = \frac{h\ A_1}{A}. \quad (F)$$

$$I = \frac{1}{4\ A}(A_1 A_2 (2\ h')^2) = \frac{h'^2 . A_1 . A_2}{A}. \quad (G)$$

Introduce these values in equation 3

$$\text{Then } M_0 = m\, W\, l = \frac{f_b\ I}{y_b} = \frac{\dfrac{f_b . h'^2 . A_1 A_2}{A}}{\dfrac{h' A_1}{A}} = f_b\, h'\ A_2. \quad (4)$$

ARTICLE 300.

THE LAST OF EQUATIONS 2.—To obtain the last value of $\frac{1}{r}$, multiply $\frac{M}{E\ I}$ by $\frac{f}{E\ y_0}$, and divide by its equivalent $\frac{M_0}{E\ I_0}$.

IX. *The value of* n''. By making the proper substitutions in Equation 5, we have

$$n''\, c^2 = \int_0^x \int_0^c \frac{c}{c - x}\, d\, x^2 = c \int_0^c \log_e \frac{c}{c - x}\, d\, x.$$

Integrate by parts . $\int u\, d\, v = u\, v - \int v\, d\, u =$ the general formula.

Let $\log_e \frac{c}{c - x} = u$ and $d\, x = d\, v$.

$$\text{Then } c \int_0^c \log_e \frac{c}{c - x} = c\, x \log_e \frac{c}{c - x} - c \int_0^c \frac{x\, d\, x}{c - x}.$$

Let $c - x = z$. Then $d\, x = -\, d\, z$; $x = c - z$

$$\therefore \int_0^c \frac{x\, d\, x}{c - x} = - \int_0^c \frac{d\, z\, (c - z)}{z} = -\ c \log_e z + z \Big\}_0^c$$

$$= \Big\{ -\ c \log_e (c - x) + (c - x) \Big\}_0^c$$

$$\therefore c \int_0^c \log_e \frac{c}{c - x}\ d\, x = \Big\{ c\, x\, .\, \log_e \frac{c}{c - x} + c^2 \log_e (c - x)$$

$$- c\,(c - x) \Big\}_0^c$$

$$= c\ \Big\{ x \log_e c - x \log_e (c - x) + c \log_e (c - x) - (c - x) \Big\}_0^c$$

$$= c \left\{ x \log_e c + (c - x) \log_e (c - x) - (c - x) \right\}_0^c = c^2$$

$\therefore n'' c^2 = c^2$ or $n'' = 1$.

X. Value of $n'' c^2 = \int_0^x \int_0^c \frac{c}{\sqrt{c^2 - x^2}} = c \int_0^c d x \, . \, \sin^{-1} \frac{x}{c}$.

Then $\frac{x}{c} = \sin y$; $\qquad d x = c \cos y \, d y$

$\therefore c \int_0^c d x \sin^{-1} \frac{x}{c} = c^2 \int_0 y \cos y \, d y =$ (by parts) $c^2 \left\{ y \sin y - \int \sin y \, d y \right\}_0^c = c^2 \left\{ y \sin y + \cos y \right\}_0^c = c^2 \left\{ \sin^{-1} \frac{x}{c} \sin . \sin^{-1} \frac{x}{c} + \cos . \sin^{-1} \frac{x}{c} \right\}_0^c =$
$c^2 \left\{ \frac{x}{c} \sin^{-1} \frac{x}{c} + \sin \left(\frac{\pi}{2} - \sin^{-1} \frac{x}{c} \right) \right\}_0^c = c^2 \left(\frac{\pi}{2} - 1 \right)$

$\therefore n'' = \frac{\pi}{2} - 1$.

Article 307.

Remark.—Near the bottom of page 334, read, m'' is never *less* than $\frac{1}{2}$.

The last member of Equation 3 should be $\frac{n f b h^2}{m'' m l}$.

Equation 13 should be $M - M_1 = \frac{w (c^2 - x^2)}{2} - M_1 = \frac{w (c^2 - x^2)}{2} - \frac{3 w c^2}{8} = \frac{3 w c^2}{8} \left(\frac{1}{3} - \frac{4 x^2}{3 c^2} \right)$.

Article 309.

Equation 3.—Read $\int_0^{y_1} y z \, d y$ for $\int_0^{y_1} y x \, d y$.

Article 312.

Equation 7.—Read $\frac{6}{20}$ for the co-efficient of the second mem-

ber. Also for $\frac{v_1''}{v_1}$ (below 7), we should have $\frac{18}{980} = \frac{2}{109}$ nearly.

ARTICLE 314.

In the first place, suppose the external load W^1 is made up, in part, by the weight of the required beam. Then, having calculated the breadth, and found the weight B^1 of the beam, it follows that, if W^1 represents the whole load, $W^1 - B^1$ will represent the load exclusive of the beam, or $\frac{W^1}{W^1 - B^1} =$ the ratio of the gross load to the load exclusive of the beam.

Equation 1.—The student needs to bear in mind that b^1 is the breadth to bear the gross load, minus the weight of the beam, and W^1 is the external load. Since h is a fixed value in this case, we have from $M_0 = m\ W\ l = n\ f\ b\ h^2$ (see page 316), and above ratio, $W^1 : W^1 - B^1 :: b : b^1$

$$\therefore b = \frac{b^1\ W^1}{W^1 - B^1} \tag{1}$$

Equation 2.—The depths being constant, the weights of the beams are proportional to their breadths.

$$\therefore B^1 : B :: b^1 : b :: b^1 : \frac{b^1\ W^1}{W^1 - B^1} :: 1 : \frac{W^1}{W^1 - B^1} \therefore B = \frac{B^1\ W^1}{W^1 - B^1} \tag{2}$$

Equation 3.—The true gross load will be

$$W^1 + B = W^1 + \frac{B^1\ W^1}{W^1 - B^1} = \frac{W'^2}{W^1 - B^1} \tag{3}$$

ARTICLE 318.

Equation 5.— From (4)

$$\frac{d^2\,x}{d\,y^2} = -\frac{P}{E\,I}\,x;\quad \frac{2\,d\,x\,.\,d^2\,x}{d\,y^2} = -\,2\,\frac{P}{E\,I}\,x\,d\,x = d\left(\frac{d\,x}{d\,y}\right)^2$$

$$\left(\frac{d\,x}{d\,y}\right)^2 = \frac{-\,P}{E\,I}\,x^2 + C. \quad \text{When } x = a,\ \frac{d\,x}{d\,y} = 0 \therefore C = a^2\,\frac{P}{E\,I}$$

$$\therefore \left(\frac{d\,x}{d\,y}\right)^2 = \frac{P}{E\,I}\,(a^2 - x^2) \therefore \frac{d\,x}{d\,y} = \sqrt{\frac{P}{E\,I}}\,\sqrt{a^2 - x^2}$$

$$\frac{d\,x}{\sqrt{a^2 - x^2}} = d\,y\,\sqrt{\frac{\mathrm{P}}{\mathrm{E\,I}}} = \frac{d\,y}{c}\ \left(\text{when } c = \sqrt{\frac{\mathrm{E\,I}}{\mathrm{P}}}\right)$$

$$\therefore \frac{d\left(\frac{x}{a}\right)}{\sqrt{1 = \frac{x^2}{a}}} = \frac{d\,y}{c}\ \therefore\ \sin^{-1}\frac{x}{a} = \frac{y}{c} + \mathrm{C}.$$

When $x = 0$, $y = 0$ $\therefore$ $c = 0$, and the above equation becomes

$$\sin^{-1}\frac{x}{a} = \frac{y}{c} \text{ or } \frac{x}{a} = \sin\frac{y}{c}\ \therefore\ x = a\sin\frac{y}{c}.$$

STRENGTH AND DEFLECTION

OF

HORIZONTAL BEAMS.

NOTATION.

l = Length of beam.
c = $\frac{1}{2}$ the length of a beam supported at both ends, or the length of a beam fixed at one end, and free at the other.
b = Breadth of beam.
h = Depth of beam.
y_0 = Distance from neutral axis to outer fiber of beam.
w = Load per unit of length of beam.
W = Total load.
R = Force at right hand support.
L = Force at left hand support.
M = Moment of external forces, at any cross-section.
M_0 = Maximum value of M.
F = Shearing force at any cross-section.
F^0 = Maximum value of F.
I = Moment of inertia of cross-section, about its neutral axis.
E = Modulus of elasticity.
f = Intensity of stress at the outer fiber.
y_1 = Maximum deflection of beam.
m = A constant factor depending upon the method of loading and supporting.
n = A constant factor depending upon the form of the beam.
k = Coëfficient of maximum deflection.
O = The Origin of a system of rectangular coördinates, in which the axis of the undeflected beam is taken as the axis of X.

The following notes on the strength and deflection of beams, contain the substance of what is found in the A. M., from Arts. 288 to 308, inclusive, and may be substituted for those articles.

It is believed that, from the simple demonstration of the fundamental formulæ for the restricted case of horizontal beams with parallel vertical loads, the average student will proceed to their application more rapidly and with greater independence and security than by following the Applied Mechanics. These pages are therefore intended to be complete in themselves.

Rankine's notation has been followed as far as practicable, and whenever introduced the letters stand for the same quantities as in the A. M.

All the notation will be found explained on page 46, or in connection with the subject where it occurs.

In all problems relating to the strength of materials, two distinct systems of forces must be considered.

1st. The external forces, consisting of the applied loads, and the supporting forces.

2d. The stresses, or forces acting amongst the fibers or particles of the material.

As these two systems of forces stand related to each other as cause and effect, it follows that, so long as equilibrium exists, the two systems must balance each other.

If we conceive of a cross-section made at any point of a beam, we can imagine the beam broken by the particles sliding upon each other, along that section (shearing), or by the pulling apart or crushing of the fibers in a direction perpendicular to the section (cross breaking). Since a part of the fibers are extended and part compressed, these stresses, of opposite kinds, form a moment about the neutral axis of the section which tends to prevent the beam from bending, and which is equal to the moment of the external forces, or "bending moment," M.

To find the value of F, take the algebraic sum of all the external forces acting upon the portion of the beam included between the section and one end, considering the upward or supporting forces as positive, and downward forces or loads upon the beam as negative.

M is found by taking the moment of all the external forces acting on the beam between the section and one end, about the neutral axis of that section. We have found it convenient in most cases to use right hand rotation as negative, and left hand rotation as positive.

The maximum values (F_0) of F, and (M_0) of M, are generally required. The former is evidently some part of the whole load (W), and the latter some portion of the product of the load and length of the beam, and is expressed by the equation $M_0 = mWl$.

Beams of sufficient strength to resist the bending moment, have usually ample strength to resist the shearing force; hence any further consideration of it here, will be omitted. (See Art. 309 A. M., and Table II in the Appendix.)

To determine the manner in which the beam resists the bending moment M, notice that when the beam is slightly curved, the layers of fibers upon the convex side are extended, while those upon the concave side are compressed, so that there will be one surface in which the fibers are neither extended or compressed.

This is called the "neutral surface." It is assumed, (which is near enough to the truth for most practical cases,) that this surface passes through the center of gravity of the cross-section of the beam, and contains the neutral axis of that section.

Within the limits of proof stress, (see Art. 245 A. M.,) the coefficients of extension and compression are exactly or nearly equal. From Hooke's law that the strains are proportional to the stresses producing them, we have zero as the stress at the neutral surface, and at any other point it is proportioned to the distance of that point from the neutral surface.

SECTION 1.—*Moment of Bending Stress.*

Let $p =$ the intensity of stress at any distance y, from OZ the neutral axis of the section.

Let $f =$ intensity of stress at outer fiber.

$d\,y\,.\,d\,z =$ an elementary area.

$p\,.\,dy\,.\,dz =$ stress on an elementary area.

$p\,.\,y\,.\,dy\,.\,dz =$ moment of this stress about the neutral axis.

$$\int_{-z}^{+z}\int_{-y_0}^{+y_0} p\,.\,y\,.\,dy\,.\,dz\,. = \text{the total moment.}$$

Since the stress is zero at the neutral surface, and increases to f at the outer edge, and is a uniformly varying stress, we have

$$p : f :: y : y_0 \text{ or } p = \frac{f}{y_0}\,y.$$

This value introduced in the above integral gives,

$$\int_{-z}^{+z}\int_{-y_0}^{+y_0} \frac{f}{y_0}\,y^2\,dy\,.\,dz = \frac{f}{y_0}\,\text{I} \quad \text{(A)},$$

which is the moment of bending stress at any section. A few values of I are given in a table at the end of Art. 95, A. M.

From these values it will be seen that $\text{I} = a\,b\,h^3$, in which a is a constant, depending upon the form of section.

Also, that $y_0 \doteq d \times h$, where d is a constant, depending upon the position of the neutral axis of the section.

$$\therefore \frac{f\,\text{I}}{y_0} = \frac{f\,a\,b\,h^3}{d\,h} = n\,f\,b\,h^2, \text{ where } n \text{ is a constant and } = \frac{a}{d}.$$

Hence the general formula for the moment of bending stress at any section is,

$$\frac{f\,\text{I}}{y_0} \text{ or } n\,f\,b\,h^2 \quad \text{(B)}.$$

TABLE 1.

VALUES OF	n
Rectangle $b\,h$, including square, ...	$\frac{1}{6}$
Hollow Rectangle, $b\,h$ on outside, and $b_{,}\,h_{,}$ on inside, ...	$\frac{1}{6}\left(1-\frac{b_{,}h_{,}^{3}}{b\,h^{3}}\right)$
Circle and Ellipse, ...	$\frac{\pi}{32}$
Hollow Ellipse, outside diameters b and h, inside $b_{,}$ and $h_{,}$, ...	$\frac{\pi}{32}\left(1-\frac{b_{1}h_{,}^{3}}{b\,h^{3}}\right)$
Hollow Circle, diameters h and $h_{,}$	$\frac{\pi}{32}\left(1-\frac{h_{,}^{4}}{h^{4}}\right)$

For other forms of cross-section, calculate n from the formula

$$n = \frac{I}{y_0\,b\,h^2}.$$

SECTION 2.—*Transverse Strength of Beams of Uniform Section.*

Calculations upon the transverse strength of beams are based upon the principle of equality of moments. The maximum moment of the external forces must be equated to the moment of bending stress at the cross-section of the beam.

Thus

$$\left\{\begin{array}{c} M_0 \text{ or } mWl \\ \text{Mo. of external forces} \end{array}\right\} = \left\{\begin{array}{c} \dfrac{f\,I}{y_0} = n\,f\,b\,h^2 \\ \text{Mo. of internal stress} \end{array}\right\} \quad (6).$$

PROBLEM I.—*To find the safe load for a given beam.*

The modulus of rupture of ash being 12,000 pounds per square inch, find the safe load that may be uniformly distrib-

uted over a rectangular ash beam whose cross-section $b \times h$ is $3'' \times 12''$, and length $l = 20\ ft.$, when supported at both ends.

As the modulus of rupture is usually given in pounds per square inch (see Table IV in the appendix of A. M.), the cross-section and length of the beam should have their dimensions expressed in inches.

Solution $l = 20 \times 12 = 240''$

The whole load, $W = wl = 2wc = 20 \times 12 \times w = 240w$. The supporting forces R and L will each be $wc = 120w$.

Let the origin O be at the center of the beam, and take a section at a distance x to the right of O. Then the moment at that section will be $M = R(c - x) - w(c - x)\left(\frac{c - x}{2}\right) = R(c - x) - \frac{w}{2}\left(c - x\right)^2$.

The maximum value of M will be obtained by making $x = 0$.

Or $M_0 = Rc - \frac{wc^2}{2} = wc^2 - \frac{wc^2}{2} = \frac{wc^2}{2} = \frac{1}{8} W l = 30\ W$.

By Art. 247 A. M., it will be seen that Rankine considered 10 as the proper factor of safety for timber, so that the outer fiber should not be under a stress greater than $12000 \div 10 = 1200$ lbs. per square inch.

$\therefore f = 1200.\ I = \frac{bh^3}{12}$ Substitute in the equation $M_0 = \frac{fI}{y_0}$ and we have

$$30W = \frac{1200 \times \frac{3 \times 1728}{12}}{\frac{12}{2}} = 200 \times 3 \times 144.$$

$W = 2880$ lbs., which is $\frac{2880}{20} = 144$ lbs. per lineal foot of beam. Ans.

Problem II.—*Knowing the form of cross-section of a given beam, and the load, to find its dimensions.*

If the modulus of rupture of cast iron beams is 40,000 lbs., find the cross-section of a rectangular beam 25 feet long, and

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Note.—The student must continually have in mind, that all weights or measures must be reduced to units of the same denomination.

Note.—The subject of factors of safety is explained in Arts. 245-247, A. M.

whose breadth and depth shall have the ratio $\frac{b}{h} = \frac{1}{3}$, for a load of 200 lbs. per lineal foot, and a concentrated load $P = 2200$ lbs. at the center, the beam being supported at both ends.

$l = 25 \times 12 = 300''$. $c = \frac{l}{2} = 150''$. $W = (25 \times 200) + 2200 = 7200$ lbs. $w = \frac{25 \times 200}{25 \times 12} = \frac{50}{3}$. {Factor of safety $= 6$} $f = 40000 \div 6 = \frac{20000}{3}$; $n = \frac{1}{6}$ (See Table I). $M_0 = n f b h^2 = n f \left(\frac{b}{h}\right) h^3$; $R = L = 3600$.

Let the origin O, be at the center of the beam, and take a section at a distance x to the right of O. Then we have

$$M = R(c - x) - \frac{w}{2}(c - x)^2$$

$$\therefore M_0 = Rc - \frac{w}{2} c^2 = 3600 \times 150 - \frac{1}{2} \frac{50}{3} \times 150^2 = 150 (3600 - 1250) = 352500.$$

$$\therefore 352500 = \frac{1}{6} \times \frac{20000}{3} \times \frac{1}{3} \times h^3 \text{ (From } M_0 = n \times f \times \frac{b}{h} \times h^3\text{)}$$

$$h = \sqrt[3]{\frac{3525 \times 27}{100}} = \text{nearly } 9.84''.$$

$$b = \frac{h}{3} = 3.28'' \text{ nearly.}$$

PROBLEM III.

Find the strongest rectangular beam that can be obtained from a given cylindrical beam.

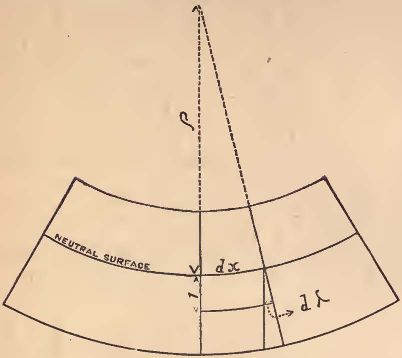
Let D be the diameter of the circular section of the given beam. We have from the equation of the circle, $D^2 = b^2 + h^2$ $\therefore h^2 = D^2 - b^2$. Since for the transverse strength of beams $M_0 = n f b h^2$, we have $M_0 = n f (D^2 b - b^3)$.

The value of b which makes M_0 a maximum is found to be,

$$b = D \sqrt{\frac{1}{3}}.$$

The corresponding value of h is, $h = D \sqrt{\frac{2}{3}}$.

Section 3.—*Equation of the Elastic Curve for Beams.*



Resume the equation $M = \frac{f\,I}{y_0}$ Let s equal the stress on a fiber at a unit's distance from the neutral axis. Then $s : f :: 1 : y_0$ or $\frac{f}{y_0} = s$

$\therefore \frac{f\,I}{y_0} = s\,I.$ (1)

Let λ be the elongation of a fiber at a unit's distance from the neutral axis. Then $d\,\lambda$ will be the elongation of a fiber whose original length was $d\,x$.

Let ρ be the radius of curvature of the neutral ~~axis~~ surface. Let E be the modulus of elasticity, that is, the force that will elongate a bar whose section is unity, to double its original length, provided the elasticity of the material does not change.

From Hooke's law, "As the stresses, so the strains," we have

$$s : E :: d\,\lambda : dx \text{ or } s = \frac{d\,\lambda}{d\,x}\,E.$$

$$\therefore \frac{f\,I}{y_0} = \frac{d\,\lambda}{d\,x}\,E\,I. \dots\dots\dots\dots\dots\dots \quad (2)$$

From the similar triangles in the figure above, we have

$$d.\,\lambda : d\,x :: 1 : \rho \;\therefore \frac{d\,\lambda}{d\,x} = \frac{1}{\rho}$$

$$\therefore \frac{f\,I}{y_0} = \frac{E\,I}{\rho} \dots\dots\dots\dots\dots\dots\dots\dots \quad (3)$$

$$\text{But } \rho = \frac{(d\,x^2 + d\,y^2)^{\frac{3}{2}}}{d\,x\,d^2\,y} = \frac{\left(1 + \frac{d\,y^2}{d\,x^2}\right)^{\frac{3}{2}}}{\frac{d^2\,y}{d\,x^2}}.$$

Since $\frac{d\,y}{d\,x}$ is the tangent of the angle that a tangent line to the curve makes with the axis of x, and in beams is exceedingly small when compared with unity, it may be neglected. Then we have

$$\rho = \frac{1}{\frac{d^2 y}{d\,x^2}} \therefore \frac{1}{\rho} = \frac{d^2 y}{d\,x^2}$$

$$\therefore \frac{f\,I}{y_0} = E\,I\frac{d^2 y}{d\,x^2} \text{ or } M = E\,I\,\frac{d^2 y}{d\,x^2} \text{ (E) which}$$

is the equation sought.

PROBLEMS.

1. Find the maximum deflection of a beam loaded uniformly, and supported at the ends.

Solution. (Origin at center.)

$$E\,I\,\frac{d^2 y}{d\,x^2} = M = wc\,(c-x) - w\,(c-x)\,\frac{(c-x)}{2}.$$

$$= w\,c\,(c-x) - \frac{w}{2}\,(c-x)^2.$$

$$E\,I\,\frac{d\,y}{d\,x} = -\frac{w\,c}{2}\,(c-x)^2 + \frac{w}{6}\,(c-x)^3 + C.$$

$$\text{When } x = 0, \frac{d\,y}{d\,x} = 0 \therefore C = \frac{w\,c^3}{2} - \frac{w\,c^3}{6} = \frac{w\,c^3}{3}.$$

$$\therefore E\,I\,\frac{d\,y}{d\,x} = -\frac{w\,c}{2}\,(c-x)^2 + \frac{w}{6}\,(c-x)^3 + \frac{w\,c^3}{3}$$

$$E\,I\,y = \frac{w\,c}{6}\,(c-x)^3 - \frac{w}{24}\,(c-x)^4 + \frac{w\,c^3\,x}{3} + D.$$

$$\text{When } x = c, y = 0 \therefore D = -\frac{w\,c^4}{3}.$$

$$\therefore E\,I\,y = \frac{w\,c}{6}\,(c-x)^3 - \frac{w}{24}\,(c-x)^4 + \frac{w\,c^3\,x}{3} - \frac{w\,c^4}{3}.$$

At the point of maximum deflection $\frac{d\,y}{d\,x} = 0$, or the curve is horizontal. In this case it is obviously at the center, where $x = 0$.

$$\therefore E\,I\,y_1 = w\,c^4\left(\frac{1}{6} - \frac{1}{24} - \frac{1}{3}\right) = -\frac{5}{24}\,w\,c^4 = -\frac{5}{384}\,W\,l^3$$

NOTE.—All the beams in this and the following section are supposed to be of uniform cross-section, horizontal in position, and with vertically applied loads.

$$\therefore y_1 = -\frac{5}{384}\frac{W l^3}{E I}. \quad \text{Ans.}$$

2. Find the maximum deflection of a beam supported at the ends, and loaded at the center with P. Ans. $\frac{1}{48}\frac{P l^3}{E I}$

3. Find the maximum deflection of a beam supported at the ends, and loaded at the center with P, and uniformly with w.

4. Find the maximum deflection of a beam fixed at one end, and loaded at the free end with P. Ans. $\frac{1}{3}\frac{P l^3}{E I}$.

5. Find the maximum deflection of a beam fixed at one end, and loaded at the free end with P, and uniformly with w.

6. Find the values of M_0 and y_1, for a beam fixed at its ends, and loaded uniformly with w, and at the center, with P. Find also the point of contra-flexure.

Let $w\, l = W$,

$$E I \frac{d^2 y}{d x^2} = M = \left(\frac{P}{2} + w\, c\right)(c - x) - \frac{w}{2}(c - x)^2 - \mu. \quad (1)$$

(from formula E.) where μ is the moment required to hold either end of the beam horizontal.

$$E I \frac{d y}{d x} = -\frac{1}{2}\left(\frac{P}{2} + w\, c\right)(c - x)^2 + \frac{w}{6}(c - x)^3 - \mu\, x + C$$

When $x = 0$, $\frac{d y}{d x} = 0 \therefore C = \left(\frac{P}{4} + \frac{w\, c}{2}\right) c^2 - \frac{w\, c^3}{6} = \frac{P\, c^2}{4} + \frac{w\, c^3}{3}$

When $x = c$, $\frac{d y}{d x} = 0 \therefore \mu\, c = C$ or $\mu = \frac{C}{c} = \frac{P\, c}{4} + \frac{w\, c^2}{3} \quad (2)$

$$\therefore E I \frac{d y}{d x} = -\frac{1}{2}\left(\frac{P}{2} + w\, c\right)(c - x)^2 + \frac{w}{6}(c - x)^3 - \left(\frac{P\, c}{4} + \frac{w\, c^2}{3}\right) x + \frac{P\, c^2}{4} + \frac{w\, c^3}{3} \quad (3)$$

$$= -\frac{1}{2}\left(\frac{P}{2} + w\, c\right)(c - x)^2 + \frac{w}{6}(c - x)^3 + \left(\frac{P\, c}{4} + \frac{w\, c^2}{3}\right)(c - x)$$

$$E I y = \frac{1}{6}\left(\frac{P}{2} + w\, c\right)(c - x)^3 - \frac{w}{24}(c - x)^4 - \frac{1}{2}\left(\frac{P\, c}{4} + \frac{w\, c^2}{3}\right)(c - x)^2 + D.$$

When $x = c$, $y = 0 \therefore D = 0$.
y is a maximum when $x = 0$.

NOTE.—"Fixed," means fixed in direction.

$$\therefore \text{E I } y_1 = \frac{\text{P } c^3}{12} + \frac{w\, c^4}{6} - \frac{w\, c^4}{24} - \frac{\text{P } c^3}{8} - \frac{w\, c^4}{6} = -\left(\frac{\text{P } c^3}{24} + \frac{w\, c^4}{24}\right)$$

$$= -\left(\frac{\text{P } l^3}{192} + \frac{w\, l^4}{384}\right) = -\frac{2\text{ P} + \text{W}_{\prime}}{384} l^3.$$

To find the value of M_0, substitute the value of μ, from equation 2, in 1

$$\text{E I } \frac{d^2 y}{d x^2} = \text{M} = \left(\frac{\text{P}}{2} + w\, c\right)(c - x) - \frac{w}{2}(c - x)^2 - \left(\frac{\text{P}c}{4} + \frac{w\, c^2}{3}\right). \qquad (4)$$

Which gives a maximum value for M, where $x = c$,

$$\therefore \text{M}_0 = -\left(\frac{\text{P } c}{4} + \frac{w\, c^2}{3}\right) = -\left(\frac{\text{P } l}{8} + \frac{w\, l^2}{12}\right) = -\left(\frac{3\text{ P} + 2\text{ W}_{\prime}}{24}\right) l.$$

To find the point of contra-flexure, make $\frac{d^2 y}{d x^2} = 0$, in equation 4, and we have

$$(c - x)^2 - 2\left(\frac{\text{P}}{2\, w} + c\right)(c - x) = -\frac{\text{P } c}{2\, w} - \frac{2\, c^2}{3}.$$

$$c - x = \frac{\text{P}}{2\, w} + c \pm \sqrt{\frac{\text{P}^2}{4\, w^2} + \frac{\text{P } c}{w} + c^2 - \frac{\text{P } c}{2\, w} - \frac{2\, c^2}{3}} = \frac{\text{P}}{2\, w} + c$$

$$\pm \sqrt{\frac{\text{P}^2}{4\, w^2} + \frac{\text{P } c}{2\, w} + \frac{c^2}{3}}.$$

$$= \frac{\text{P } l}{2\text{W}_{\prime}} + \frac{l}{2} \pm \sqrt{\frac{\text{P}^2\, l^2}{4\text{ W}_{\prime}^{\,2}} + \frac{\text{P } l^2}{4\text{W}_{\prime}} + \frac{l^2}{12}}.$$

$$= l\left(\frac{\text{P}}{2\text{W}_{\prime}} + \frac{1}{2} \pm \sqrt{\frac{3\text{ P}^2 + 3\text{ P W}_{\prime} + \text{W}_{\prime}^{\,2}}{12\text{ W}_{\prime}^{\,2}}}\right)$$

$$= l\left(\frac{\text{P}}{2\text{W}_{\prime}} + \frac{1}{2} \pm \frac{1}{2\text{W}_{\prime}}\sqrt{\frac{(3\text{ P} + \text{W}_{\prime})^2}{3} - 2\text{ P}^2 - \text{P W}_{\prime}}\right).$$

= distance from end.

7. Find the value of R; the point of Max. deflection ($x_{\prime}$); the Max. bending moment; and the point of inflection for a beam fixed at L, supported at R, and loaded uniformly.

$\text{R} = \frac{3}{8}\, w\, l = \frac{3}{8}\text{ W}$,
$x_{\prime} = .58\, l$ (nearly.) from L,
Max. moment $\frac{1}{8}\text{ W } l$,
Point of inflection at $\frac{1}{4}\, l$ from L.

8. Find the point of contra-flexure, and values of R, y_1, and M_0, for a beam loaded at the center with P, fixed at L, and supported at R.

First consider that part of the curve included between the right hand support and the center.

$$E\,I\,\frac{d^2 y}{d\,x^2} = R\,(2\,c - x) = M \ldots\ldots\ldots\ldots \quad (1)$$

$$E\,I\,\frac{d\,y}{d\,x} = -\frac{R\,(2\,c - x)^2}{2} + C \ldots\ldots\ldots \quad (2).$$

$$E\,I\,y = \frac{R}{6}\,(2\,c - x)^3 + C\,x + D \ldots\ldots\ldots (3).$$

When $x = 2\,c$, $y = 0 \therefore D = -2\,C\,c$.

$$\therefore E\,I\,y = \frac{R}{6}\,(2\,c - x)^3 - C\,(2\,c - x) \ldots\ldots \quad (4).$$

Now consider that part of the beam included between the center and the left hand support,

$$E\,I\,\frac{d^2\,y'}{d\,x'^2} = R\,(2\,c - x') - P\,(c - x') \ldots\ldots \quad (5).$$

$$E\,I\,\frac{d\,y'}{d\,x'} = -\frac{R}{2}\,(2\,c - x')^2 + \frac{P}{2}\,(c - x')^2 + F$$

When $x' = 0$, $\frac{d\,y'}{d\,x'} = 0 \therefore F = 2\,R\,c^2 - \frac{P\,c^2}{2} = \frac{c^2}{2}\,(4\,R - P)$

$$\therefore E\,I\,\frac{d\,y'}{d\,x'} = -\frac{R}{2}\,(2\,c - x')^2 + \frac{P}{2}\,(c - x')^2 + \frac{c^2}{2}\,(4\,R - P) \ldots\ldots\ldots\ldots\ldots\ldots \quad (6.)$$

$$E\,I\,y' = \frac{R}{6}\,(2\,c - x')^3 - \frac{P}{6}\,(c - x')^3 + \frac{c^2}{2}\,(4\,R - P)\,x' + G.$$

When $x' = 0$, $y' = 0 \therefore G = -\frac{8\,R\,c^3}{6} + \frac{P\,c^3}{6} = \frac{c^3}{6}\,(P - 8\,R.)$

$$\therefore E\,I\,y' = \frac{R}{6}\,(2\,c - x')^3 - \frac{P}{6}\,(c - x')^3 + \frac{c^2\,x'}{2}\,(4\,R - P) + \frac{c^3}{6}\,(P - 8\,R) \ldots\ldots\ldots\ldots \quad (7).$$

When $x = x' = c$, $y = y'$ and $\frac{d\,y}{d\,x} = \frac{d\,y'}{d\,x}$.

Under these conditions, we have from 2 and 6

$$-\frac{R\,c^2}{2} + C = -\frac{R\,c^2}{2} + \frac{c^2}{2}\,(4\,R - P)$$

$\therefore C = \frac{c^2}{2} (4 R - P)$, and equation 4 becomes

$$E I y = \frac{R}{6} (2 c - x)^3 - \frac{c^2}{2} (4 R - P) (2 c - x) \ldots (8).$$

We also obtain from the above condition and 7 and 8, $\frac{R c^3}{6} - \frac{c^3}{2} (4 R - P) = \frac{R c^3}{6} + \frac{c^3}{2} (4 R - P) + \frac{c^3}{6} (P - 8 R)$; $0 = (4 R - P) + \frac{1}{6} (P - 8 R) = 24 R - 6 P + P - 8 R \therefore R = \frac{5}{16} P \ldots\ldots (9).$

To find the point of maximum deflection, substitute the values of R and C in equation 2, and we have

$$E I \frac{d y}{d x} = \left(- \frac{5}{32} (2 c - x)^2 + \frac{c^2}{8} \right) P \ldots (10).$$

The maximum deflection is where $\frac{d y}{d x} = 0$, hence to find the point, we have $0 = - \frac{5}{32} (2 c - x)^2 + \frac{c^2}{8}$

$\therefore 5 (2 c - x)^2 = 4 c^2$

$(2 c - x) = \pm 2 c \sqrt{\frac{1}{5}}$

$x = 2 c - 2 c \sqrt{\frac{1}{5}} = l \left(1 - \sqrt{\frac{1}{5}} \right) =$ distance from fixed end. The amount of maximum deflection will be found by substituting this value of x in equation 8, which gives y_1

$$= \frac{1}{E I} \left(\frac{5}{96} P \left(l - l \left(1 - \sqrt{\tfrac{1}{5}}\right)\right)^3 - \frac{l^2}{8} \left(\frac{5 P}{4} - P\right) \left(l - l \left(1 - \sqrt{\tfrac{1}{5}}\right)\right) \right)$$

$$= \left(\frac{5}{96} \sqrt{\frac{1}{125}} - \frac{1}{32} \sqrt{\frac{1}{5}} \right) \frac{P l^3}{E I}$$

$$= \sqrt{\frac{1}{5}} \left(\frac{1}{96} - \frac{3}{96} \right) \frac{P l^3}{E I}$$

$$= - \frac{1}{48} \sqrt{\frac{1}{5}} \frac{P l^3}{E I}.$$

Substitute the value of R in equation 5.

$$\text{Then E I } \frac{d^2 y'}{d x'^2} = \frac{5}{16} \mathrm{P} (2 c - x') - \mathrm{P} (c - x') \ldots \ldots (11).$$

To find the point of contra-flexure, make $\frac{d^2 y'}{d x'^2} = 0$

$$\text{Then } \frac{5}{16} \mathrm{P} (2 c - x') - \mathrm{P} (c - x') = 0$$

$$10 c - 5 x' = 16 c - 16 x'$$

$$x' = \frac{6}{11} c = \frac{6}{22} l = \text{point of contra-flexure.}$$

To find M_0 from equation 11 we have

$$\mathrm{M} = \frac{\mathrm{P}}{16} (10 c - 5 x' - 16c + 16 x') = \frac{\mathrm{P}}{16} (11 x' - 6 c)$$

Making $x' = 0$, we have

$$\mathrm{M}_0 = - \frac{3}{8} \mathrm{P} c = - \frac{3}{16} \mathrm{P} l.$$

9. Find the same quantities for a beam fixed at one end, supported at the other, and loaded uniformly, and at the center with P.

10. Given a beam supported at the ends and center, with a fixed load over its entire length, and an additional load over one span, to find R, and the point of contra-flexure.

Let w be the intensity of the fixed load, and w_1 that of the additional load.

The tendency of this load is to curve the opposite end of the beam upwards, so that instead of requiring a support it would have to be fastened down at that end.

Let the left hand span be thus loaded, then R will act downward.

Since the fixed load is distributed symmetrically with reference to the center, in calculating for R, it may be considered as all supported at the center.

Taking the origin of coördinates at L, and the coördinates of the left hand $x_{\prime}$ and $y_{\prime}$, and those of the right hand span, x and y, we have the following:

For the latter

$$\mathrm{E\ I} \frac{d^2 y}{d x^2} = - \mathrm{R} (l - x).$$

$$\mathrm{E\ I} \frac{d y}{d x} = \frac{\mathrm{R}}{2} (l - x)^2 + \mathrm{A}.$$

$$E\,I\,y = -\frac{R}{6}(l-x)^3 + A\,x + B.$$

For the former

$$E\,I\,\frac{d^2 y_{,}}{d\,x_{,}^2} = -L\,x_{,} + \frac{w_{,}x_{,}^2}{2}.$$

$$E\,I\,\frac{d\,y_{,}}{d\,x_{,}} = -\frac{L\,x_{,}^2}{2} + \frac{w_{,}x_{,}^3}{6} + C.$$

$$E\,I\,y_{,} = -\frac{L\,x_{,}^3}{6} + \frac{w_{,}x_{,}^4}{24} + C\,x + D.$$

These equations contain six constants, A, B, C, D, R, and L. The six equations required for their determination, may be found from the following conditions:

When $x = \frac{l}{2}$, $y=0$. When $x_{,}=\frac{l}{2}$, $y_{,}=0$.

" $x = l$, $y=0$. " $x_{,}=0$, $y_{,}=0$.

" $x = x_{,} = \frac{l}{2}$, $\frac{d\,y}{d\,x} = \frac{d\,y_{,}}{d\,x_{,}}$,

" $x = x_{,} = \frac{l}{2}$, $\frac{d^2 y}{d\,x^2} = \frac{d^2 y_{,}}{d\,x_{,}^2}$.

Forming the equations and eliminating we have $R = -\frac{1}{16}\left(\frac{w_{,}l}{2}\right) = \frac{1}{16}$ the additional load. $A = -\frac{R\,l^2}{24}$.

$B = \frac{R\,l^3}{24}$; $L = \frac{7}{16}\left(\frac{w_{,}l}{2}\right) = \frac{7}{32}w_{,}l$.

To determine the point of contra-flexure, put $\frac{d^2 y_{,}}{d\,x_{,}^2}=0$, remembering that M must include the fixed load.

$$E\,I\,\frac{d^2 y_{,}}{d\,x_{,}^2} = -L\,x_{,} + (w + w_{,})\frac{x_{,}^2}{2} = M$$

$$= -\frac{7}{32}w_{,}l\,x_{,} + (w + w_{,})\frac{x_{,}^2}{2} = 0$$

$$x_{,} = \frac{7}{16}\frac{w_{,}\,l}{w+w_{,}} = \text{distance from left end.}$$

TABLE 2.

Beams of Uniform Section.	Values of $M_0 = mWl$.	Values of y_1
1. Loaded at the center with W. Supported at the ends.	$\frac{1}{4}.Wl.$	$\frac{1}{48}\frac{W\,l^3}{E\,I}.$
2. Loaded uniformly with w. Supported at the ends.	$\frac{1}{8}Wl.$	$\frac{5}{384}.\frac{W\,l^3}{E\,I}.$
3. Loaded at free end with W. Fixed at one end, free at other.	$W\,l.$	$\frac{1}{3}.\frac{W\,l^3}{E\,I}.$
4. Loaded uniformly with w. Fixed as in 3.	$\frac{1}{2}.\,W\,l.$	$\frac{1}{8}.\frac{W\,l^3}{E\,I}.$
5. Loaded with w, and at the extreme end with P. Fixed as in 3. ($w\,l = W_{,}$).	$\frac{l}{2}(2\,P + W_{,}).$	$\frac{l^3}{EI}\left(\frac{1}{3}P+\frac{1}{8}W\right)$
6. Loaded at center with W. Fixed at the ends.	$\frac{1}{8}.Wl.$	$\frac{1}{192}\frac{W\,l^3}{E\,I}.$
7. Loaded uniformly with w. Fixed at the ends.	$\frac{1}{12}.\,W\,l.$	$\frac{1}{384}.\frac{W\,l^3}{E\,I}.$
8. Loaded uniformly with w, and at the center with P. Fixed at the ends.	$\left(\frac{3\,P+2\,W_{,}}{24}\right).$	$\frac{(2\,P+W_{,})}{384}\frac{l^3}{E\,I}.$
9. Loaded uniformly with w. Fixed at one end and supported at the other.	$\frac{1}{8}\,W\,l.$	$\frac{2.08}{384}.\frac{W\,l^3}{E\,I}$ nearly.
10. Fixed as in 9, loaded at center with W.	$\frac{3}{16}.\,W\,l.$	$\frac{1}{48}\sqrt{\frac{1}{5}}\frac{W\,l^3}{E\,I}.$

NOTE.—To calculate the transverse strength of beams, use the formula, $mWl = nfbh^2$. Take the value of mWl from table 2 and the value of n from page 49.

The preceding table gives the values of Max. bending moment, and Max. deflection for the most common and important cases of loading and supporting. In the column headed $M_0 = m\ W\ l$, the co-efficients of $W\ l$ which are the values of m for the different cases, are so arranged as to be seen at a glance and readily taken out for use. Also the co-efficients of $\frac{W\ l^3}{E\ I}$ in the column headed y_1 are the values of k in the formula $y_1 = k.\ \frac{W\ l^3}{E\ I}$, or co-efficients of Max. deflection. (Cases 5 and 8 are not included in the above remarks relating to the co-efficients).

SECTION 4.—*Equation for designing a beam whose deflection shall not exceed a given amount.*

In the preceding problems we have seen that the value of y_1 may be expressed by the equation $y_1 = \frac{k\ W\ l^3}{E\ I}$ (1).

$$\text{Hence } k = \frac{y_1\ E\ I}{W\ l^3} \qquad (2).$$

From formula 6 in section 2, we have M_0 or $m\ W\ l = \frac{f\ I}{y_0}$

$\therefore\ I = \frac{m\ W\ l\ y_0}{f}$.

Substitute this value for I in equation (1).

$$y_1 = \frac{k\ W\ l^3\ f}{E\ m\ W\ l\ y_0} = \frac{k}{m} \times \frac{f}{E} \times \frac{l^2}{y_0}. \qquad (3).$$

In beams symmetrical with respect to the neutral axis of the section, $y_0 = \frac{1}{2}\ h$; hence (3) becomes $y_1 = \frac{2k}{m} \times \frac{f}{E} \times \frac{l^2}{h}$.

$$\frac{y_1}{l} = \frac{2k}{m} \times \frac{f}{E} \times \frac{l}{h}. \qquad \text{(F.)}$$

$$\text{or } \frac{h}{l} = \frac{2k}{m} \times \frac{f}{E} \times \frac{l}{y_1}. \qquad \text{(G.)}$$

In applying this formula, k and m are to be taken from table (2), or calculated as heretofore explained. E and f are found in the Appendix to the Applied Mechanics, tables 1 and 4 respectively; but the value of f there given, being the *ultimate* strength of the materials, *it must be divided by a proper factor of safety before being introduced into the formula.*

$\frac{l}{y_1}$ is the ratio of the length of the beam to the greatest deflexion allowable for the particular case in question, and is assumed from experience or from the necessary limitations of the problem.

PROBLEM.

Find the dimensions of the rectangular cross-section of a wrought iron beam 20 feet long, which shall deflect but $\frac{1}{1000}$ of its length, for a working load of 225 lbs. per lineal foot.

$$\left.\begin{array}{l} k = \frac{5}{384} \\ m = \frac{1}{8} \end{array}\right\} \text{See Table 2,} \qquad E = 27000000.$$

Working strength of wrought iron,

$$\left. n = \frac{1}{6} \right\} \text{See Table 1,} \qquad f = 9000.$$

$\therefore \frac{f}{E} = \frac{1}{3000}$; $l = 12 \times 20 = 240''$; $W = 4500$ lbs; $\frac{l}{y_1} = 1000$.

From formula G, we have

$$\frac{h}{l} = \frac{2\,k}{m} \cdot \frac{f}{E} \cdot \frac{l}{y_1} = \frac{2 \times \frac{5}{384}}{\frac{1}{8}} \times \frac{1}{3000} \times 1000 =$$

$$\frac{50}{720} = \frac{1}{14,4}.$$

From formula 6, section 2, we have $m\,W\,l = n\,f\,b\,h^2$

$$\therefore\ b = \frac{m\,W\,l}{n\,f\,h^2} = \frac{\frac{1}{8} \times 4500 \times 240}{\frac{1}{6} \times 9000 \times \frac{2500}{9}} = \frac{81}{250} \text{ inch.}$$

SECTION 5.—*Beams of uniform strength.*

CASE 1.—Any beam of uniform strength, and uniform depth, not fixed at both ends.

From the equation of the elastic curve, formula E, section 3, we have

$$E\,I\,\frac{d^2 y}{d x^2} = M \therefore E\,\frac{d^2 y}{d x^2} = \frac{M}{I} \ \ldots\ldots \ (1).$$

$M = \dfrac{f\,I}{y_0}$ or $\dfrac{M}{I} = \dfrac{f}{y_0}$ from formula 6, section 2.

Since the depth is constant, y_0 is constant $\therefore$ equation (1) becomes

$$E\,\frac{d^2 y}{d x^2} = \frac{M}{I} = \frac{f}{y_0} = \text{a constant} \ldots\ldots\ (2).$$

$$E\,\frac{d y}{d x} = \frac{f}{y_0}\,x + C.$$

Take the origin so that when $x = 0$, $\dfrac{d y}{d x} = 0$, (generally at the center or end of beam) then $C = 0$

$\therefore\ E\,y = \dfrac{f}{2\,y_0}\,x^2 + D.$ When $x = 0$, $y = 0$ $\therefore$ $D = 0.$

$\therefore\ E\,y = \dfrac{f}{2\,y_0}\,x^2$; y is a maximum when $x = c$

$\therefore\ y_1 = \dfrac{f\,c^2}{2\,E\,y_0}$ = maximum deflection. $c = l$ or $\dfrac{1}{2}\,l$ according to the method of supporting.

In section 3, we had the radius of curvature $= \rho = \dfrac{E\,I}{M}$

As E is a constant, and $\dfrac{I}{M}$ is constant as above shown for this case, then ρ is constant, or the curve is the arc of a circle.

CASE 2.—Beam of uniform strength, uniform breadth, built in at one end, free at the other, and loaded uniformly.

As before, we have $E\,\dfrac{d^2 y}{d x^2} = \dfrac{f}{y_0}.$

Let y_z = maximum half depth of beam, and h_z its depth,

Then $M_0 = \dfrac{w\,c^2}{2} = n\,f\,b\,h^2_z$

$$M = \frac{w\,(c - x)^2}{2} = n\,f\,b\,h^2$$

$$\therefore\ \frac{c^2}{(c - x)^2} = \frac{h_z^{\,2}}{h^2} = \frac{y_z^{\,2}}{y_0^{\,2}} \ \therefore\ \frac{1}{y_0} = \frac{1}{y_z}\left(\frac{c}{c - x}\right)$$

$$\therefore\ E\,\frac{d^2 y}{d x^2} = \frac{f\,c}{y_z}\left(\frac{1}{c - x}\right)$$

$$E \frac{d y}{d x} = - \frac{f c}{y_z}, \log_e (c - x) + C$$

When $x = 0$ $\frac{d y}{d x} = 0$ $\therefore$ $C = \frac{f c}{y_z} \log_e c$

$$\therefore E y = \frac{f c}{y_z} \int \log_e \left(\frac{c}{c - x}\right) d x. \quad \text{Integrate by parts.}$$

$$E y = \frac{f c}{y_z} \left\{ x \log_e \left(\frac{c}{c - x}\right) - \int \frac{x \, d x}{c - x} \right\}$$

Make $c - x = z$ and integrate.

$$E y = \frac{f c}{y_z} \left\{ x \log_e \left(\frac{c}{c - x}\right) + c \log_e (c - x) - (c - x) \right\} + F$$

When $x = 0, y = 0$ $\therefore$ $F = \frac{f c}{y_z} (- c \log_e c + c)$

$$\therefore E y = \frac{f c}{y_z} \left\{ x \log_e \left(\frac{c}{c - x}\right) + c \log_e (c - x) - (c - x) - c \log_e c + c \right\}$$

$$= \frac{f c}{y_z} \left\{ \log_e c \times (x - c) + \log_e (c - x) \times (c - x) + x \right\}$$

y is a maximum when $x = c$

$$y_1 = \frac{f c^2}{E y_z} = \frac{f l^2}{E y_z}.$$

Case 3.—A beam of uniform strength, uniform breadth, built in at one end, and loaded at the other.

$$y_1 = \frac{2 f l^2}{3 E y_z}.$$

Case 4.—A beam of uniform strength and breadth, loaded at the middle and supported at the ends.

$$y_1 = \frac{2}{3} \frac{f c^2}{E y_z}.$$

Case 5.—A beam of uniform strength and breadth, loaded uniformly, and supported at the ends.

$$y_1 = \left(\frac{\pi}{2} - 1\right) \frac{f c^2}{E y_z}.$$

Note.—In case 5 integrate the expression $\sin^{-1} \frac{x}{c} \, dx$ by making $\sin^{-1} \frac{x}{c} = z$, or $\frac{x}{c} = \sin z$, and it becomes $cz \cos z dz$. Integrate by parts and substitute the value of z.

CASE 6.—A beam of uniform strength, depth, and load, fixed at its ends.

For the plan and elevation of such a beam, see Figs. 144 and 143 of A. M.

Contra-flexure.

As the beam is of uniform strength and depth, the radius of curvature will everywhere be the same. (See Case 1.) And as it is fixed at C and A, the point of contra-flexure must be, half way between A and C. (Fig. 143 A. M.)

$$\therefore \mathrm{C\,B} = \frac{c}{2} = \frac{1}{4}\, l. \qquad (1.)$$

Maximum deflection.

It is obvious that the portion B A B is in the condition of a uniformly loaded beam, supported at its ends, whose length is $c = \frac{1}{2}\, l$. Hence, as the beam is of uniform strength and depth, its deflection will be found by Case 1.

$$y_1' = \frac{f\left(\frac{c}{2}\right)^2}{2\,\mathrm{E}\,y_0} = \frac{f\,c^2}{8\,\mathrm{E}\,y_0} = \frac{f\,l^2}{32\,\mathrm{E}\,y_0} = \text{vertical distance between}$$
A and B.

By the geometry of the figure, it will be seen that the vertical distance between C and A, is twice that between A and B.

$$\therefore y_{\prime} = \frac{f\,c^2}{4\,\mathrm{E}\,y_0} = \frac{f\,l^2}{16\,\mathrm{E}\,y_0}. \qquad (2.)$$

Moment of flexure at A.

Since the portion B A B is similar to a uniformly loaded beam supported at the ends, the supporting forces are $\frac{wc}{2}$, and the moment anywhere between B and B is $\mathrm{M} = \frac{wc}{2}\left(\frac{c}{2} - x\right) - \frac{w}{2}\left(\frac{c}{2} - x\right)^2$.

The moment at A is given by making $x = 0$,

$$\therefore \mathrm{M}_a = \frac{w\,c^2}{8} = \frac{\mathrm{W}\,c}{16} = \frac{\mathrm{W}\,l}{32}. \qquad (3.)$$

Moment at C.

From B to C the beam is in the condition of one fixed at one end, loaded at the other with $\frac{wc}{2}$, and uniformly with w. Hence the greatest moment of flexure is at C, where $M_c = \left(\frac{w\,c}{2}\right) \frac{c}{2} + \left(\frac{w\,c}{2}\right) \frac{c}{4} = \frac{3\,w\,c^2}{8} = \frac{3\,W\,c}{16} = \frac{3\,W\,l}{32}$. (4.)

Comparison of widths at A *and* C.

From equations 3 and 4, we see that the moment at C, is three times its value A. Hence, as the depth is constant, the breadth at the end must theoretically be three times the center breadth.

See theoretical plan of the beam, fig. 144 A. M.

Moment of flexure anywhere.

For the general expression for the moment of flexure, we have

$$M_x = w\,c\,(c - x) - \frac{w}{2}\,(c - x)^2 - \mu,$$

where μ is the moment required to hold the ends horizontal; hence it is equal to the moment at C $= \frac{3\,W\,c}{16} = \frac{3\,W\,l}{32}$ see equation 4. $\therefore$

$$M_x = w\,c\,(c - x) - \frac{w}{2}\,(c - x)^2 - \frac{3\,W\,c}{16}.$$

$$= \frac{w}{2}\,(c^2 - x^2) - \frac{3\,W\,c}{16}.$$

$$= \frac{W}{2\,l}\left(\frac{l^2}{4} - x^2\right) - \frac{3\,W\,l}{32}.$$

$$= W\,l\left(\frac{1}{32} - \frac{x^2}{2\,l^2}\right). \qquad (5.)$$

= the general equation of the moment of flexure for any point in the beam.

ARTICLE 365.

Equation 1.—The following method of deducing the formula for the centrifugal force of a unit of mass leads to Equation 1, and is a good exercise for the student in the application of some fundamental principles of dynamics:

Let R = the resultant accelerating force acting on a unit of mass. Let the path of this particle be referred to three rectangular axes X, Y, and Z.

Then since the force is measured by the product of mass and acceleration, and the mass = 1, we have

$$\frac{d^2 x}{d t^2} = \text{component of R along X},$$

$$\frac{d^2 y}{d t^2} = \text{component of R along Y},$$

$$\frac{d^2 z}{d t^2} = \text{component of R along Z},$$

$$\therefore R^2 = \left(\frac{d^2 x}{d t^2}\right)^2 + \left(\frac{d^2 y}{d t^2}\right)^2 + \left(\frac{d^2 z}{d t^2}\right)^2 - \left(\frac{d^2 s}{d t^2}\right)^2 + \left(\frac{d^2 s}{d t^2}\right)^2$$ (subtracting and adding $\left(\frac{d^2 s}{d t^2}\right)^2$).

The square of the radius of curvature of any curve is

$$\rho^2 = \frac{\left(\frac{d s^2}{d t^2}\right)^2}{\left(\frac{d^2 x}{d t^2}\right)^2 + \left(\frac{d^2 y}{d t^2}\right)^2 + \left(\frac{d^2 z}{d t^2}\right)^2 - \left(\frac{d^2 s}{d t^2}\right)^2}$$ (See Church's Calculus, page 146, Equation 2.)

Substituting the value of the denominator of the second member of this equation for the same quantity in the value of R^2 above, we have $R^2 = \frac{\left(\frac{d s^2}{d t^2}\right)^2}{\rho^2} + \left(\frac{d^2 s}{d t^2}\right)^2 = \left(\frac{v^2}{\rho}\right)^2 + \left(\frac{d^2 s}{d t^2}\right)^2$. Since R^2 = the sum of the squares of two quantities, they must be the rectangular components of R; and as one of these, $\left(\frac{d^2 s}{d t^2}\right)$ is the force in the direction of the mo-

tion at any instant, the other $\left(\frac{v^2}{\rho}\right)$ must be the deviating force = centrifugal force.

Article 390.

Equations 1 *and* 2.—Since the radius vector is normal to the curve, it coincides in direction with the radius of curvature, and both are shifted through the same angle $d\,i$ in the time $d\,t$. The arc described by the point A, Fig. 187, A. M., is measured by $\rho\,d\,i$; or since $c\,r$ is the velocity of A, by $c\,r\,d\,t$. Hence, Equation (1) $\rho\,d\,i = c\,r\,d\,t$.

The distance through which the end T of the radius vector is carried in the time $d\,t$, owing to the revolution about C, is $b\,r_2\,d\,t$, measured along the arc, whose center is at C. Its projection on a perpendicular to A T is $b\,r_2\,d\,t\cos\theta$, which is the arc subtending the angle through which r has been shifted negatively. This arc divided by r gives the angle

$$-\frac{b\,r_2\,d\,t\cos\theta}{r};\ d\,i = \frac{c\,r\,d\,t}{r} - \frac{b\,r_2\,d\,t\cos\theta}{r} = d\,t\left(c - \frac{b\,r_2\cos\theta}{r}\right).$$

Article 402.

Read $\frac{n}{2}$ instead of n, in the paragraph preceding Eq. (6).

Article 421.

Equation 2.—Read $\frac{d\,.\,Q\,\rho}{d\,s} = \frac{d\,.\,A\,u\,\rho}{d\,s}$.

Article 422.

Equation (1).—The last member should read $\frac{\rho\,d\,v}{d\,t}$.

Article 423.

Equation 2.—The second parenthesis of the first member should be $\left(u\,\frac{d\,\rho}{d\,x} + v\,\frac{d\,\rho}{d\,y} + w\,\frac{d\,\rho}{d\,z} + \frac{d\,\rho}{d\,t}\right)$.

Article 443.

The eccentricity of an ellipse is the distance from one focus to the center, divided by the transverse axis, or $\frac{\sqrt{a^2 - b^2}}{a}$.

ARTICLE 444.

$\overline{\text{X Y}} = \sqrt{(\overline{\text{G H}}^2 + \overline{\text{X W}}^2)}$. A section of the hyperboloid through X Y, and perpendicular to A B, Fig. 195, A. M., would be a circle of radius X Y, and the projection of the generating line in the position F E, on the plane of this section would be a vertical line at a distance G H from the center of the section. The point W is where this line cuts the circle, and its distance from the axis of the hyperboloid is evidently

$\text{X Y} = (\overline{\text{G H}}^2 + \overline{\text{X W}}^2)^{\frac{1}{2}}$. Let the student draw the section as above directed.

ARTICLE 450.

Equation (1). The velocity of P_1 in a plane perpendicular to C_1 is $a_1\ \overline{C_1 P_1}$. Since the line of action makes the angle i with the axis c_1, it makes $(90° - i_1)$ with the direction of the motion of P_1. The component of the velocity of P_1 along the line of action, will, therefore, be $a_1\ \overline{C_1 P_1} \cos (90° - i_1) = a_1\ \overline{C_1 P_1}\ \sin i_1$. By the same reasoning the velocity of P_2 along the line of action is $a_2\ \overline{C_2 P_2} \sin i_2$, and these are equal by the principle stated in this article. Hence Equation (1).

ARTICLE 458.

Equation (1). r is the length of cord unwound from the circle whose radius is $C_1\ P_1$ (Fig. 198), while a distance q on the circle whose radius is C_1 I passes the point I. These arcs are proportional to the radii of their respective circles, or

$$\frac{r}{q} = \frac{C_1\,P_1}{C_1\,I} = \sin \theta.$$

ARTICLE 462.

Equation (3). Substituting values from Equations (1) and (2) of this article in the equation $s = 2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \int_0^{q_1} r\,d\,q$ we have $s = 4\ r_0 \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \int_0^{2r_0\left(\frac{\pi}{2} - \theta\right)} \sin \frac{q}{2\ r_0}\ d\ q$. Multiply and divide the second member by $2\ r_0$, and we have $s = 8\ r_0^2$

$$\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\int_0^{2r_0}\left(\frac{\pi}{2}-\theta\right)\sin\frac{q}{2\,r_0}\,d\,\frac{q}{2\,r_0} = 8r_0^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$
$$\left(-\cos\left(\frac{\pi}{2}-\theta\right)+1\right) = 8\,r_0^2\left(\frac{1}{r_1}+\frac{1}{r_2}\right)(1-\sin\theta).$$

Article 463.

See Willis' Principles of Mechanism, page 137.

Article 482.

The degree of approximation attained by putting $\sqrt{c^2-(r_1-r_2)^2} = c - \frac{(r_1-r_2)^2}{2\,c}$ is seen by squaring both members of the *inequality*. The term $\frac{(r_1-r_2)^4}{(2\,c)^2}$ is neglected in the second member.

Article 483.

Equation (1).—The first value of $y = r_0 - \frac{r_1+r_2}{2}$ is obvious from an inspection of the figure of the conoid. To obtain the equation $r_0 - \frac{r_1+r_2}{2} = \frac{(r_1-r_2)^2}{2\,\pi\,c}$ it is to be considered that the length of belt for two equal pulleys of radius r_0 is $L = 2\,c + 2\,\pi\,r_0$, which, placed equal to the approximate value of L from Equation (3 A), Article 482, gives $2\,c + 2\,\pi\,r_0 = 2\,c + \pi\,(r_1+r_2) + \frac{(r_1-r_2)^2}{c}$ or $r_1 + r_2 = 2\,r_0 - \frac{(r_1-r_2)^2}{\pi\,c}$ which is also Equation (2) of this article. The manner of using this equation for designing a pair of stepped cones in which two opposite pulleys are of equal radii is explained in the A. M.

If it is required to complete a pair of cones when the radii of a pair of opposite pulleys are given, the value of r_0 may first be found by substituting the given values for r_1 and r_2, in equation (2).

Then replacing r_0 by this value, the radius of one of the required pulleys may be assumed and the radius of its opposite pulley calculated by solving the resulting quadratic equation.

Problem.—In a pair of stepped cones let the radii of the end pulleys opposite each other be $r_1 = 24''$, $r_2 = 6''$, and let the distance of their centers be $c = 48''$ required the radii of the

adjacent pair of pulleys on the stepped cones. From (2) we have

$$2\,r_0 = (r_1 + r_2) + \frac{(r_1 - r_2)^2}{\pi\,c} = 30 + \frac{(18)^2}{3\frac{1}{7} \times 48''}\ (\pi = 3\tfrac{1}{7}\ \text{nearly})$$

$$= 32.15\ \text{nearly.} \quad \therefore\ r_1 + r_2 = 32.15 - \frac{(r_1 - r_2)^2}{\pi\,c}.$$

Let r_1 and r_2 now represent respectively the radii of the greater and smaller of the pulleys to be next calculated.

We may now fix the value of r_1 and calculate r_2. It will however generally be easier to find r_2 by trial than to solve the quadratic equation.

Let r_1 be fixed at 20″, and for the first trial let us suppose r_2 to be equal to 11.8″. Then $r_1 + r_2 = 31.8$, $r_1 - r_2 = 8.2''$, and we have from the formula $31.8 = 32.15 - \dfrac{(8.2)^2 \cdot 7}{1056} = 31.7$ nearly.

Again let $r_2 = 11.6''$ for a second trial, then $31.6 = 32.15 - .468 = 31.68$. Thus we see that the true value of r_2 lies between 11.6″ and 11.8″; $r_2 = 11.7''$ very nearly satisfies the equation.

ARTICLE 488.

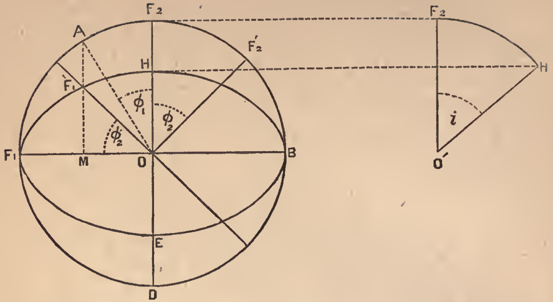
In Fig. 218 let $C_2\,T_2 = r$, $T_1\,T_2 = l$, the angle $T_2\,C_2\,T_1 = \alpha$, and the angle $T_2\,T_1\,C_2 = \beta$. Then $C_2\,A = (r \cos \alpha + l \cos \beta) \tan\,\beta$.

Placing the above value of $C_2\,A$ in equation (2), and eliminating β by means of the equation, $l \sin \beta = r \sin \alpha$, we have v_1

$$= v_2 \sin \alpha \left\{ \frac{\frac{r}{l} \cos \alpha}{\sqrt{1 - \frac{r^2}{l_2} \sin^2 \alpha}} + 1 \right\}$$ from which the velocity of the piston of an engine may be calculated for any given crank angle; for by knowing the number of revolutions, the value of v_2 is easily found.

ARTICLE 491.

Equations (2) and (1).—Let the axis C_2 be perpendicular to the plane of the paper and let the plane of the axes of the shafts be vertical. The projection on the paper, of the path of F_2 will be the circle $F_1\,F_2\,B\,D$, and the projection of the path of F_1 will be the ellipse $F_1\,H\,B\,E$ of the accompanying figure.



Suppose F_2 to move to the position F_2', F_1 will in the same time move to F_1'. The angle $F_2\,O\,F_2'$ is ϕ_2. The angle $F_1\,O\,F_1'$ which is also equal to ϕ_2, is the projection of the angle described by $O\,F_1$ in the plane of its path. The true magnitude of this angle is $F_1\,O\,A$, (found by revolving the plane of the ellipse in which the angle lies, about $F_1\,B$ until the ellipse coincides with the circle, when F_1 will fall at A in the vertical line $M\,A$.) The angle $F_2\,O\,A$ is ϕ_1. (See the A. M.)

Then $O\,M \cot \phi_1 = M\,A$. $\quad O\,M \tan \phi_2 = M\,F_1'$.

$$\therefore \frac{\tan \phi_2}{\cot \phi_1} = \frac{M\,F_1'}{M\,A} = \frac{O\,H}{O\,F_2} = \cos i \text{ or } \tan \phi_1 \tan \phi_2 = \cos i.$$

Differentiating this equation we have

$$-\frac{d\,\phi_2}{d\,\phi_1} = \frac{\tan \phi_2 \cos^2 \phi_2}{\tan \phi_1 \cos^2 \phi_1} = \frac{\tan \phi_2\,(1 + \tan^2 \phi_1)}{\tan \phi_1\,(1 + \tan^2 \phi_2)}.$$

$$= \frac{\cot \phi_1 \left(1 + \dfrac{1}{\cot^2 \phi_1}\right)}{\cot \phi_2 \left(1 + \dfrac{1}{\cot^2 \phi_2}\right)} = \frac{\tan \phi_1 + \cot \phi_1}{\tan \phi_2 + \cot \phi_2}.$$

$$\therefore \frac{a_2}{a_1} = \frac{\tan \phi_1 + \cot \phi_1}{\tan \phi_2 + \cot \phi_2}.$$ Eliminate ϕ_2 by Equation (1),

$$\tan \phi_2 = \frac{\cos i}{\tan \phi_1}.$$

Then $$\frac{a_2}{a_1} = \frac{\cos i\,(1 + \tan^2 \phi_1)}{\tan^2 \phi_1 + \cos^2 i}.$$

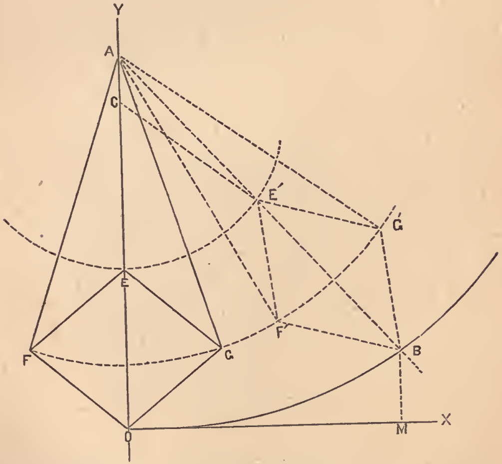
When F_1 is in the plane of the axes $\phi_1 = 0$, $\tan \phi_1 = 0$ and $\frac{a_2}{a_1} = \frac{1}{\cos i}$, which is the first of Equations (1) A. M. In a simi-

lar manner by eliminating ϕ_1, the second of Equations (1) may be obtained. When $\frac{a_2}{a_1} = \frac{\cos i\,(1 + \tan^2 \phi_1)}{\tan^2 \phi_1 + \cos^2 i} = 1$ or the shafts have equal angular velocities, we find $\tan^2 \phi_1 = \cos i$; $\phi_1 = \tan^{-1}. \sqrt{\cos i}$. (See Willis' Principles of Mechanism, Article 513.)

ARTICLE 507.

Equations (3).—The square of the perpendicular, drawn from any point in the circumference of a circle to a diameter, is equal to the product of the segments into which the foot of the perpendicular divides the diameter $\therefore \left(\frac{s}{2}\right)^2 = \mathrm{T\,V}\,(2\,r - \mathrm{T\,V}) = \mathrm{T\,V}\,.\,2\,\mathrm{C\,M}.$

An exact parallel motion, accomplished entirely by means of link work (without any sliding piece, as in Watts' Exact Parallel



Motion, Article 507, A. M.), has recently been discovered by M. Peaucillier, an outline of which is here given.

In the accompanying figure, O F E G is an equilateral parallelogram of bars, jointed at each of its corners. A and C are fixed points; A F and A G are links jointed at their extremities. C E is also a link jointed to the corner (E) of the paralelogram, and capable of turning about C. When the point O is moved in the plane of the figure, the points F and G must describe arcs of circles about A. E must describe an arc about C.

When this combination of links is made to oscillate about the fixed points A and C:

1st. If C is midway between A and E, the point O will be guided in a straight line perpendicular to A O.

2d. If C is in any other position on the line A E, the point O will move in the arc of a circle.

To prove these propositions, and also to find exact relations between the radius of the arc described by O. and the position of C on the line A E, is the object of the following analytical investigation:

Suppose the instrument to be swung aside until the axis A O takes the position A B, making an angle θ with its neutral position. E will be at E′ on this line, and also on the circle whose center is at C. The parallelogram will be partially closed as represented by the dotted lines in the figure. O will have moved to some point B on the new position of the axis, and it is required to find the locus of this point.

Let O be the origin, O X the horizontal, and O Y the vertical axis. Then O M $= x$, M B $= y$.

Let A F $=$ A G $= l$; F E $=$ E G $= s$; A E′ $= c$; A O $= k$; C E $= r$; C A $= d$, and the angle F′ E′ B $=$ F′ B E′ $= \beta$.

Then $x = (k - y) \tan \theta$ (1)

$$y = k - (c + 2\, s \cos \beta) \cos \theta \qquad (2)$$

$$l^2 = c^2 + s^2 + 2\, c\, s \cos \beta \qquad (3)$$

$$\text{A E}' = c = d \cos \theta + \sqrt{r^2 - d^2 \sin^2 \theta} \qquad (4)$$

Combining 2 and 3 we have

$$y = k - \left(c + \frac{l^2 - c^2 - s^2}{c}\right) \cos \theta = k - \frac{l^2 - s^2}{c} \cos \theta \qquad (5)$$

Combining this with (4)

$$y = k - \frac{(l^2 - s^2) \cos \theta}{d \cos \theta + \sqrt{r^2 - d^2 \sin^2 \theta}}. \qquad (6)$$

Putting θ in terms of $\tan \theta$ gives

$$y = k - \frac{\dfrac{l^2 - s^2}{\sqrt{1 + \tan^2\theta}}}{\dfrac{d}{\sqrt{1 + \tan^2\theta}} + \sqrt{r^2 - \dfrac{d^2 \tan^2\theta}{1 + \tan^2\theta}}}$$

$$= k - \frac{l^2 - s^2}{d + \sqrt{r^2 + (r^2 - d^2)\tan^2\theta}}$$

Substituting θ from (1), transposing k and changing the signs of both members

$$k - y = \frac{(k - y)(l^2 - s^2)}{d(k - y) + \sqrt{r^2 (k - y)^2 + x^2 (r^2 - d^2)}}.$$

Dropping the factor $(k - y)$, clearing of radicals, and placing $(l^2 - s^2) = n$, we have $(r^2 - d^2)(k - y)^2 + (r^2 - d^2) x^2 = n^2 - 2nd(k - y)$ (7) $(r^2 - d^2) k^2 - 2k(r^2 - d^2) y + (r^2 - d^2) y^2 + (r^2 - d^2) x^2 = n^2 - 2ndk + 2ndy$, which is apparently the equation of an ellipse (since $b^2 - 4ac < 0$. See Church's Analytical Geometry, page 207).

Since the center of this curve is evidently on the line O A produced, which contains one of the principal axes, let us transfer the origin of coördinates to some point on this line at a distance a from O. The formulæ for such transformation are $x = x'$; $y = y' + a$. Then from (7) $(r^2 - d^2)(k - a - y')^2 + (r^2 - d^2) x'^2 = n^2 - 2nd(k - a - y')$.

Let $k - a = b$, and drop the primes; $(r^2 - d^2) x^2 + (r^2 - d^2) y^2 = n^2 - 2ndb - (r^2 - d^2) b^2 + \{ 2nd + 2(r^2 - d^2) b \} y$. If the origin is at the center of the curve, the term containing y must disappear or $2nd + 2(r^2 - d^2) b = 0$; $b = \dfrac{nd}{r^2 - d^2}$.

Replacing b by its value, we have $k - a = -\dfrac{nd}{r^2 - d^2}$

$$a = k + \frac{nd}{r^2 - d^2}.$$

Since the co-efficients of x^2 and y^2 are equal, the ellipse has equal axis, or is a circle of radius $a = k + \dfrac{nd}{r^2 - d^2}$. When $d = r$, $a = \infty$, or the locus is a straight line Q . E . D .

ARTICLE 515.

PROBLEM.—Find the formula for the work performed upon the piston of a steam engine during one stroke, the length of

stroke being l, the area of the piston a, the initial pressure of the steam p_1, the steam being cut off at a distance c from the beginning of the stroke.

$$\text{Ans. } a\, p_1\, c\left(1 + \log_e \frac{l}{c}\right).$$

ARTICLE 531.

Equation (1).—Let $g' =$ the force of gravity at any point on the earth's surface at the level of the sea.

Let $T =$ time of one revolution of the earth on its axis.

Let $n =$ number of revolutions in a unit of time.

Then $n\,T = 1$.

The centrifugal force of a unit of mass at the level of the sea and latitude λ is

$$F = \frac{v^2}{R\cos\lambda} = \frac{(2\,\pi\,.\,R\cos\lambda\,.\,n)^2}{R\cos\lambda} = \frac{4\,\pi^2\,R}{T^2}\cos\lambda.$$

The component of F acting opposite to gravity is $f = F\cos\lambda$

$$= \frac{4\,\pi^2\,R}{T^2}\cos^2\lambda = \frac{2\,\pi^2\,R}{T^2}(1 + \cos 2\,\lambda).$$

Gravity being diminished by this amount, if we subtract it from G the real attraction of the earth, we shall have

$$g' = G - \frac{2\,\pi^2\,R}{T^2} - \frac{2\,\pi^2\,R}{T^2}\cos .\, 2\,\lambda.$$

The value of g' for latitude 45° has been found by experiment. Denote this value by g_1.

Then from the formula (making $\lambda = 45°$)

$$g_1 = G - \frac{2\,\pi^2\,R}{T^2}. \qquad \therefore g' = g_1 - \frac{2\,\pi^2\,R}{T^2}\cos(2\lambda).$$

$$= g_1\left(1 - \frac{2\,\pi^2\,R}{g_1\,T^2}\cos 2\,\lambda\right).$$

For a point at a distance h above the earth's surface (h being small compared with R) we have from Newton's law of gravity

$$\frac{g'}{g} = \frac{\left(\frac{1}{R}\right)^2}{\left(\frac{1}{R+h}\right)^2}. \qquad \therefore g = g'\frac{R^2}{(R+h)^2} = g'\frac{R^2}{R^2 + 2\,R\,h + h^2}.$$

Performing the division to two places

$$g = g'\left(1 - \frac{2\,h}{R}\right) \therefore g = g_1\left(1 - \frac{2\,\pi^2\,R}{g_1\,T^2}\cos 2\,\lambda\right)\left(1 - \frac{2h}{R}\right).$$

ARTICLE 537.

Equation (2)—Let a be the angle which a tangent to the curve makes at any point.

$$\text{Then } Q = W \cos a = \frac{W}{\sqrt{1+\tan^2 a}} = \frac{W}{\sqrt{1+\frac{(dz)^2}{(dx)^2}}}.$$

Since the horizontal component of the velocity of the projectile is constant we have

$$v_0 \cos\theta = v \cos a. \quad \therefore \cos a = \frac{v_0 \cos\theta}{v}.$$

$$\therefore Q = W . \frac{v_0 \cos\theta}{v}.$$

ARTICLE 545.

Equation (1)—The component of g urging the pendulum towards its lowest point is $g \sin\theta$ in which θ is the inclination of the curve to the horizon. To show that in the cycloidal pendulum this force is proportional to the length of the arc between any position of the pendulum and its lowest position, we consider that the length of any curve is $s = \int ds = \int \rho\, d\theta$.

From Art. 390 A. M., we have for the cycloid $\rho = 4\, r_2 \cos\theta$.

$$\therefore s = 4\, r_2 \int_0^\theta \cos\theta\, d\theta = 4\, r_2 \sin\theta \text{ which varies as } \sin\theta.$$

Hence the vibrations of a cycloidal pendulum are isochronous.

ARTICLE 552.

PROBLEM.—A body weighing 100 lbs. starts from a point 48 feet above the earth's surface. After falling freely 16 feet it encounters a constant resistance equal to $\frac{3}{4}$ gravity. Required its potential energy before starting; its actual energy, when at a distance of 16 feet from the earth; its potential energy at the same point; the sum of its actual and potential energies when it reaches the earth; also the work done in the fall.

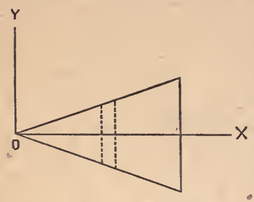
ARTICLE 557.

Equation 6 should read as follows:

$$\frac{W v^2}{2g} + \frac{Q x}{2} = \frac{W v_0^2}{2g} \sin^2 at + \frac{Q_1 x_1}{2} \cos^2 at = \frac{Q_1 x_1}{2}.$$

ARTICLE 578.

A method of finding the moment of inertia of solids of revolution is illustrated in the solution of the following problem. Find the moment of inertia of a cone whose altitude is h and radius of base r, about an axis through its vertex and perpendicular to its geometric axis. Conceive a portion of the cone included between two planes at right angles to its axis, and at a distance dx apart. Find the moment of inertia of this elementary slice or disc, by applying to *it* the theorem of Art. 576 A. M.



This will give $\frac{w\,\pi\,y^4}{4}\,dx\,.+w\,\pi\,y^2\,dx\,.\,x^2.$

$$\therefore\ \mathrm{I}=w\frac{\pi}{4}\int_o^h y^4\,dx+w\,\pi\int_o^h y^2\,x^2\,dx=\frac{w\,\pi\,r^2\,h}{20}\left(r^2+4\,h^2\right).$$

If the moment of inertia about the axis O X were required, the axis of inertia would pass through the center of gravity of the disc and be perpendicular to its bases. We should therefore have $\mathrm{I}_x=\frac{w\,\pi}{2}\int_o^h y^4\,dx.$ This is the formula for problem I of this Article.

PROBLEM III.—Although the ellipsoid is not a solid of revolution, its moment of inertia may be found by the method above explained.

Let the origin be at the center of the ellipsoid and let O X, O Y and OZ be in the direction of the semi-axes a, b and c respectively. A section perpendicular to O X is an ellipse whose semi-axes are y and z. The moment of inertia of the elementary slice with respect to O X is therefore

$\frac{w\,\pi\,y^3\,z}{4}\,dx+\frac{w\,\pi\,y\,z^3}{4}\,dx.$ (See problem in Notes on Art. 95.)

$$\therefore\ \mathrm{I}=\frac{w\,\pi}{4}\int_{-a}^{a}y^3\,z\,dx+\frac{w\,\pi}{4}\int_{-a}^{a}y\,z^3\,dx.$$

The values of z and y are taken from the following equations.

$$\begin{cases} a^2\,z^2+c^2\,x^2=a^2\,c^2\\ a^2\,y^2+b^2\,x^2=a^2\,b^2\end{cases}.$$

$$\therefore \mathrm{I} = w \frac{\pi\, b^3 c}{4\, a^4} \int_{-a}^{a} (a^2 - x^2)^2\, dx + \frac{w\, \pi\, b\, c^3}{4\, a^4} \int_{-a}^{a} dx\, (a^2 - x^2)^2$$
$$= \frac{4\, w\, \pi\, a\, b\, c\, (b^2 + c^2)}{15}.$$

Let the student perform the same problem by integrating the following formula according to the directions in Note on Art. 83, assigning the proper limits.

$$\mathrm{I} = w \int \int \int dx\, dy\, dz\, (y^2 + z^2).$$

Theorem.—The actual energy of a body revolving about a fixed axis is found by multiplying its moment of inertia by the square of its angular velocity and dividing the product by twice the force of gravity.

Let ρ = the radius of gyration of the body, a its angular velocity, and W its weight. If the mass of the body were concentrated at the center of gyration, (that is, at a distance ρ from the axis of rotation) its velocity would be $a\,\rho$. Its actual energy would be $\frac{\mathrm{W}}{2\,g}\, a^2\, \rho^2 = \frac{a^2}{2\,g} \cdot \mathrm{W}\, \rho^2 = \frac{a^2}{2\,g}\, \mathrm{I}.$

If the axis of rotation traverses the center of gravity of the body and is a "line of symmetry of the figure of the body," the values of I and ρ may be taken from the columns headed I_0 and ρ_0 in the table of Art. 578 A. M. (See Art. 589 A. M.)

Article 579.

Read $\frac{b^2 + c^2}{6}$ instead of $\frac{b^2 + c}{6}$ under example.

Article 590.

Equation 5.—From equation (3) of Art. 585, we have $s^2 = \frac{1}{\mathrm{I}}$.

From equation (6) of Art. 588 we have $\cos\theta = \frac{n}{s} = \frac{\mathrm{I}}{\sqrt{\mathrm{I}^2 + \mathrm{K}^2}}$.

$\therefore s = \frac{n\,\sqrt{\mathrm{I}^2 + \mathrm{K}^2}}{\mathrm{I}}. \qquad \therefore s^2 = \frac{1}{\mathrm{I}} = \frac{n^2\,(\mathrm{I}^2 + \mathrm{K}^2)}{\mathrm{I}^2}.$

Or $1 = \frac{n^2}{\mathrm{I}}(\mathrm{I}^2 + \mathrm{K}^2) \cdot \therefore \frac{\mathrm{I}}{n^2} = \mathrm{I}^2 + \mathrm{K}^2 = \mathrm{I}_1^2 \cos^2 a + \mathrm{I}_2^2 \cos^2 \beta + \mathrm{I}_3^2 \cos^2 \gamma.$ Equation (2)

NOTE.—The moment of inertia as here used is defined in Arts. 571, 572, and 573, A. M.

But $I = I_1 \cos^2 \alpha + I_2 \cos^2 \beta + I_3 \cos^2 \gamma$. (Eq. (3), Art. 585)

$\therefore \frac{I_1 \cos^2 \alpha}{n^2} + \frac{I_2 \cos^2 \beta}{n^2} + \frac{I_3 \cos^2 \gamma}{n^2} = I_1^2 \cos^2 \alpha + I_2^2 \cos^2 \beta + I_3^2$ $\cos^2 \gamma$.

Or $\left(I_1^2 - \frac{I_1}{n^2}\right) \cos^2 \alpha + \left(I_2^2 - \frac{I_2}{n^2}\right) \cos^2 \beta + \left(I_3^2 - \frac{I_3}{n^2}\right)$ $\cos^2 \gamma = 0$.

ARTICLE 606.

First Equation.—The moment of the deviating couple in terms of E is $M = 2\ E \tan \alpha$. The value of the constant E is the sum of its components with reference to the axes C G and C E, and these components are $\frac{a^2\ I_1}{2\ g} \cos^2 \alpha$ and $\frac{a^2\ I_2}{2\ g} \cos^2$ $(180 - (90^\circ - \alpha)) = - \frac{a^2\ I_2}{2\ g} \sin^2 \alpha \quad \therefore M = \frac{a^2}{g} (I_1 - I_2)$ $\cos \alpha \sin \alpha = \frac{W}{g} a^2 (\rho_1^2 - \rho_2^2) \cos \alpha \sin \alpha$.

ARTICLE 638.

Formulæ for computing the discharge and diameter of pipes, are given in Article 450 of Rankine's Civil Engineering.

ARTICLE 648.

Equations (1), (2), and (3).—These equations may be obtained as follows: The velocity of the jet before it meets the surface is v. When the surface has changed its direction through an angle β, the component of its velocity in the original direction of the jet is $v \cos \beta$. The change of velocity is, therefore, $v - v \cos \beta = v (1 - \cos \beta)$. Hence, the change of momentum which measures the force in this direction, is $F_x = \frac{\rho\ Q}{g} v$ $(1 - \cos \beta)$. The change of velocity in a direction perpendicular the original direction of the jet is $v \sin \beta \quad \therefore F_y = \frac{\rho\ Q}{g} v \sin \beta$. The resultant $F = \sqrt{F_x^2 + F_y^2} = \frac{\rho\ Q\ v}{g}$ $\sqrt{(1 - \cos \beta)^2 + \sin^2 \beta} = \frac{\rho\ Q}{g} v \sqrt{2\ (1 - \cos \beta)} = \frac{\rho\ Q}{g} v$ $\sqrt{4 \sin^2 \frac{\beta}{2}} = \frac{2\ \rho\ Q}{g} v \sin \frac{\beta}{2}$.

Art. 532. Page 486.

Eq. (3) in Art. 531 will be as follows.

$m = \frac{W}{g}$. (3) $m = \frac{W \times 32 \text{ poundals}}{g}$ = $gm = W \times 32$ poundals

$32 \times m = W \times 32$ poundals. from which we get. $m = W$.

The unit of Mass = 1 lb. (16 oz)

The Wt. of 1 pound = 32.2 poundals.